Integration By Parts

■ The Tabular Method

Sometimes it is necessary to apply the method of integration by parts several times before we obtain an integral we can evaluate. In the case where the integral is of the form

$$\int p(x)f(x)\,dx$$

where $p(x)$ is a polynomial, the method below describes a useful way of organizing the results of iterated integration by parts.

Tabular Method

(1) Differentiate $p(x)$ repeatedly until you obtain 0 and list the results in the first column.
(2) Integrate $f(x)$ repeatedly and list the results in the second column.
(3) Draw an arrow from each entry in the first column to the entry one row down in the second column.
(4) Label the arrows with alternating $+$ and $-$ signs, starting with $+$.
(5) For each arrow, form the product of the expressions at its tip and tail and multiply by -1 if the arrow is labeled $-$. Add up these products to obtain the value of the integral.

▶ **Example 1** Use tabular integration by parts to evaluate the integral $\int 4x^4 \sin 2x\,dx$.

Solution We make the following table:

derivatives of $4x^4$		antiderivatives of $\sin 2x$
$4x^4$	$+$	$\sin 2x$
$16x^3$	$-$	$-\frac{1}{2}\cos 2x$
$48x^2$	$+$	$-\frac{1}{4}\sin 2x$
$96x$	$-$	$\frac{1}{8}\cos 2x$
96	$+$	$\frac{1}{16}\sin 2x$
0		$-\frac{1}{32}\cos 2x$

Then by the tabular method we have that

$$\int 4x^4 \sin 2x\,dx = -2x^4\cos 2x + 4x^3\sin 2x + 6x^2\cos 2x - 6x\sin 2x - 3\cos 2x + C$$

■ Definite Integrals

Using the fundamental theorem of calculus we can extend the method of integration by parts to definite integrals. This gives the formula

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$$

▶ **Example 2** Evaluate the definite integral $\displaystyle\int_2^4 \sec^{-1}\sqrt{x}\,dx$.

Solution We choose

$$u = \sec^{-1}\sqrt{x} \qquad\qquad dv = dx$$

and from these we find

$$du = \frac{1}{2x\sqrt{x-1}}\,dx \qquad\qquad v = x$$

Then we have

$$\begin{aligned}
\int_2^4 \sec^{-1}\sqrt{x}\,dx &= \Big[x\sec^{-1}\sqrt{x}\Big]_2^4 - \int_2^4 \frac{1}{2\sqrt{x-1}}\,dx \\
&= \left(4\sec^{-1}\sqrt{4} - 2\sec^{-1}\sqrt{2}\right) - \Big[\sqrt{x-1}\Big]_2^4 \\
&= \left(4\sec^{-1}2 - 2\sec^{-1}\sqrt{2}\right) - \left(\sqrt{3}-1\right) \\
&= 4(\pi/3) - 2(\pi/4) - \sqrt{3} + 1 \\
&= 5\pi/6 - \sqrt{3} + 1
\end{aligned}$$