Integration By Parts

■ The Product Rule

Let f and g be functions and suppose that G is an antiderivative for g. This means that $G'(x) = g(x)$. The product rule states that

$$\frac{d}{dx}[f(x)G(x)] = f(x)g(x) + f'(x)G(x).$$

We can rewrite this rule in integral notation as

$$\int [f(x)g(x) + f'(x)G(x)]\,dx = f(x)G(x).$$

Distributing the integral over the sum we get

$$\int f(x)g(x)\,dx + \int f'(x)G(x)\,dx = f(x)G(x)$$

and moving the second integral to other side of the equation gives the formula

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx.$$

The usefulness of this formula is that if the integral on the left is too difficult, we can replace it with the expression on the right, which contains a different integral, one that might be easier to evaluate. This method is called **integration by parts**.

This formula can be rewritten as follows. Let $u = f(x)$. Then $du = f'(x)\,dx$. Also let $v = G(x)$. Then $dv = G'(x)\,dx = g(x)\,dx$. Then the formula above becomes

$$\int u\,dv = uv - \int v\,du$$

▶ **Example 1** Use integration by parts to evaluate the integral

$$\int x \ln x\,dx.$$

Solution Keep in mind when choosing u and dv that we will need to integrate $v\,du$. With this in mind choose

$$u = \ln x$$
$$dv = x\,dx$$

Then differentiating u and integrating dv we get

$$du = \frac{1}{x}\,dx$$
$$v = \frac{x^2}{2}$$

Then by the formula we have

$$\int x \ln x\,dx = (\ln x)\left(\frac{x^2}{2}\right) - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \frac{1}{2}\int x\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C$$

Making the correct choice for u and dv comes with practice, but sometimes the following method is helpful. If we want to integrate $f(x)g(x)$ and f and g are functions that fall into different categories in the list:

(1) **L**ogarithmic

(2) **I**nverse trigonometric

(3) **A**lgebraic

(4) **T**rigonometric

(5) **E**xponential

If the category of f comes before the category of g then set $u = f(x)$ and $dv = g(x)\,dx$. A helpful mnemonic device might be

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▶ **Example 2** Use integration by parts to evaluate the integral

$$\int x \sin 3x \, dx$$

Solution Using the above strategy, since x is algebraic and $sin3x$ is trigonometric we should choose

$$\begin{aligned} u &= x \\ dv &= \sin 3x \, dx \end{aligned}$$

Then differentiating u and integrating dv we get

$$\begin{aligned} du &= dx \\ v &= -\frac{1}{3}\cos 3x \end{aligned}$$

Then by the integration by parts formula we have

$$\begin{aligned} \int x \sin 3x \, dx &= (x)\left(-\frac{1}{3}\cos 3x\right) - \int -\frac{1}{3}\cos 3x \, dx \\ &= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x \, dx \\ &= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C \end{aligned}$$

■ Repeated Applications

Sometimes it is necessary to use integration by parts several times before we obtain an integral that is easy enough to evaluate.

▶ **Example 3** Use repeated integration by parts to evaluate the integral

$$\int x^2 e^x \, dx$$

Solution Using the L.I.A.T.E. strategy, since x^2 is algebraic and e^x is exponential we should choose

$$u = x^2$$
$$dv = e^x \, dx$$

Then differentiating u and integrating dv we get

$$du = 2x \, dx$$
$$v = e^x$$

Then by the integration by parts formula we have

$$\int x^2 e^x \, dx = (x^2)(e^x) - \int 2x e^x \, dx$$
$$= (x^2)(e^x) - 2 \int x e^x \, dx \qquad (*)$$

This isn't quite enough since we still can't integrate xe^x directly. But we can do this integral by parts. Now using the L.I.A.T.E. strategy on $\int xe^x \, dx$ we choose

$$u = x$$
$$dv = e^x dx$$

Then differentiating u and integrating dv we get

$$du = dx$$
$$v = e^x$$

By the formula for integration by parts we have

$$\int xe^x = xe^x - \int e^x \, dx = xe^x - e^x$$

Now we can substitute this expression back into (*) to get

$$\int x^2 e^x \, dx = (x^2)(e^x) - 2 \int x e^x \, dx = x^2 e^x - 2\left(xe^x - e^x\right) + C$$