Monogenic $S_4$ Quartic Fields Arising from Elliptic Curves

Joint work with Kate Stange and Alden Gassert

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Background
Let $K$ be a number field. We say $K$ is **monogenic** if the ring of integers $\mathcal{O}_K$ admits a power $\mathbb{Z}$-basis. That is, if there is some monic, irreducible $f(x) \in \mathbb{Z}[x]$ with a root $\theta$ such that $\mathcal{O}_K$ has a $\mathbb{Z}$-basis $\{1, \theta, \ldots, \theta^{n-1}\}$, then $K$ is monogenic.
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**Examples:** Quadratic fields. The field $\mathbb{Q}(\sqrt{d})$ has a ring of integers with $\mathbb{Z}$-basis $\left\{1, \frac{1 + \sqrt{d}}{2}\right\}$ if $d \equiv 1$ modulo 4 and $\{1, \sqrt{d}\}$ otherwise.
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Cyclotomic fields. The ring of integers $\mathcal{O}_{\mathbb{Q}(\zeta_p)}$ has $\mathbb{Z}$-basis $\left\{1, \zeta_p, \ldots, \zeta_p^{p-2}\right\}$. 
Consider $x^3 - x^2 - 2x - 8$ and let $\theta$ be a root. Dedekind showed $\mathbb{Q}(\theta)$ is not monogenic.
Consider $x^3 - x^2 - 2x - 8$ and let $\theta$ be a root. Dedekind showed $\mathbb{Q}(\theta)$ is \textit{not} monogenic.

A “good” $\mathbb{Z}$ basis is

$$\left\{ 1, \frac{1}{2}(\theta + \theta^2), \theta^2 \right\}.$$
For the rest of the talk let $E$ be an elliptic curve over $\mathbb{Q}$. If $P = (x, y) \in E(\mathbb{Q})$ then we can describe the multiplication by $m$ map quite explicitly:

$$[m]P = \left( \frac{\phi_m(P)}{\Psi_m(P)^2}, \frac{\omega_m(P)}{\Psi_m(P)^3} \right)$$

where $\phi_m, \Psi_m, \omega_m \in \mathbb{Z}[x, y]$. If $m$ is odd then $\Psi_m \in \mathbb{Z}[x]$. 
Division Polynomials and Partial Torsion Fields

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Generally, $\mathbb{Q}(E[m])$ is called the $m^{th}$ torsion field or $m^{th}$ division field. If $m$ is odd and $\psi_m$ is irreducible we define the $m^{th}$ partial torsion field to be the extension of $\mathbb{Q}$ obtained by a root of $\psi_m$. 
Another Definition of Division Polynomials

If we write

\[ E : \quad y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6, \]

then we can define \( \Psi_n \) recursively starting with

\[ \Psi_1 = 1, \]
\[ \Psi_2 = 2y + a_1 x + a_3, \]
\[ \Psi_3 = 3x^4 + b_2 x^3 + 3b_4 x^2 + 3b_6 x + b_8, \]
\[ \frac{\Psi_4}{\Psi_2} = 2x^2 + b_2 x^5 + 5b_4 x^4 + 10b_6 x^3 + 10b_8 x^2 + (b_2 b_8 - b_4 b_6) x + (b_4 b_8 - b_6^2), \]

and using the formulas

\[
\begin{align*}
\Psi_{2m+1} &= \Psi_{m+2} \Psi_m^3 - \Psi_{m-1} \Psi_m^3 \quad \text{for } m \geq 2, \\
\Psi_{2m+1} \Psi_2 &= \Psi_{m-1}^2 \Psi_m \Psi_{m+2} - \Psi_{m-2} \Psi_m \Psi_{m+1}^2 \quad \text{for } m \geq 3.
\end{align*}
\]
The Main Result and Context
Theorem

Suppose that $\alpha \pm 8$ are squarefree, where $\alpha \in \mathbb{Z}$. Let $\theta$ be a root of the irreducible polynomial $T^4 - 6T^2 - \alpha T - 3$. Then the ring of integers of $\mathbb{Q}(\theta)$ has $\mathbb{Z}$-basis $\{1, \theta, \theta^2, \theta^3\}$. That is, $\mathbb{Q}(\theta)$ is a monogenic quartic field. Moreover, $\mathbb{Q}(\theta)$ has Galois group $S_4$. 
Our Main Result

Theorem

Let $E$ be an elliptic curve defined over $\mathbb{Q}$, such that some twist $E'$ of $E$ has a 4-torsion point defined over $\mathbb{Q}$. Then the following are equivalent:

1. $E'$ has reduction types $I^*$ and $I_1$ only;
2. $E$ has $j$-invariant with squarefree denominator except a possible factor of 4.
3. $E$ has $j$-invariant $j = (\alpha^2 - 48)^3 (\alpha - 8)(\alpha + 8)$, where $\alpha \in \mathbb{Z}$, $\alpha \pm 8$ are squarefree.

Let $\theta$ be a root of $\Psi_3$. If any of the above hypotheses holds, then the third partial torsion field, $\mathbb{Q}(\theta)$, is monogenic with a generator given by a root of $T^4 - 6T^2 - \alpha T - 3$. Note the generator of the power basis is not $\theta$. Moreover, $\mathbb{Q}(\theta)$ has discriminant $-27(\alpha - 8)^2(\alpha + 8)^2$.\[7\]
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1. $E'$ has reduction types $l_1^*$ and $l_1$ only;
2. $E$ has $j$-invariant with squarefree denominator except a possible factor of 4.
3. $E$ has $j$-invariant $j = \frac{(\alpha^2 - 48)^3}{(\alpha - 8)(\alpha + 8)}$, where $\alpha \in \mathbb{Z}$, $\alpha \pm 8$ are squarefree.

Let $\theta$ be a root of $\Psi_3$. If any of the above hypotheses holds, then the third partial torsion field, $\mathbb{Q}(\theta)$, is monogenic with a generator given by a root of $T^4 - 6T^2 - \alpha T - 3$. Note the generator of the power basis is not $\theta$. Moreover, $\mathbb{Q}(\theta)$ has discriminant $-27(\alpha - 8)^2(\alpha + 8)^2$. 
Often, if you want to look at elliptic curves with 4-torsion over $\mathbb{Q}$, you look at the curve

$$E : y^2 + (\alpha + 8\beta)xy + \beta(\alpha + 8\beta)^2y = x^3 + \beta(\alpha + 8\beta)x^2.$$ 

This parametrization is called Tate’s normal form.
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$$\Psi_3 = 3x^4 + ((\alpha + 8\beta)^2 + 4\beta(\alpha + 8\beta))x^3 + 3\beta(\alpha + 8\beta)^3x^2 + 3\beta^2(\alpha + 8\beta)^4x + \beta^3(\alpha + 8\beta)^5.$$
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\]

However, we found the model that worked best for us was the Fueter form:

\[
T_1^2 = 4T^3 + \frac{\alpha}{\beta} T^2 + 4T.
\]

Here the identity is \((0, 0)\) and \((1, \sqrt{8 + \frac{\alpha}{\beta}})\) is a point of order 4.
Recall

\[ \Psi_3 = 3x^4 + ((\alpha + 8\beta)^2 + 4\beta(\alpha + 8\beta))x^3 + 3\beta(\alpha + 8\beta)^3x^2 + 3\beta^2(\alpha + 8\beta)^4x + \beta^3(\alpha + 8\beta)^5. \]
Why $T^4 - 6T^2 - \alpha T - 3$?

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When we change to the Fueter form $\Psi_3$ becomes $T^4 - 6T^2 - \frac{\alpha}{\beta}T - 3$. 
Hasse's problem: When is $\mathbb{Q}(\theta)$ monogenic?

Torsion fields, $\mathbb{Q}(E[m])$, are a generalization of cyclotomic fields. Cyclotomic fields and their maximal real subfields are monogenic. We have class field theory. Cyclotomic fields have very nice formulas for ramification and for the field discriminant.

Torsion on Elliptic Curves in general: Given a Galois group or a degree, what torsion subgroups can elliptic curves over number fields with that Galois group or that degree have?
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Torsion on Elliptic Curves in general: Given a Galois group or a degree, what torsion subgroups can elliptic curves over number fields with that Galois group or that degree have?
Proof Ideas
Let \( f(x) \) be a monic, irreducible, integer polynomial with root \( \theta \) and let \( p \) be a prime. The Montes algorithm takes in \( f(x) \) and through successive reductions and expansions tells us about \( v_p \left( \mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta] \right) \).
Let $f(x)$ be a monic, irreducible, integer polynomial with root $\theta$ and let $p$ be a prime. The Montes algorithm takes in $f(x)$ and through successive reductions and expansions tells us about $v_p\left([\mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta]]\right)$.

Recall,

$$\text{disc} \left(\mathbb{Q}(\theta)\right) \cdot [\mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta]]^2 = \text{disc}(f).$$
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Recall,

$$\text{disc} (\mathbb{Q}(\theta)) \cdot \left[ \mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta] \right]^2 = \text{disc}(f).$$

In particular,

$$v_p \left( \text{disc}(\mathbb{Q}(\theta)) \right) + 2v_p \left( \left[ \mathcal{O}_{\mathbb{Q}(\theta)} : \mathbb{Z}[\theta] \right] \right) = v_p \left( \text{disc}(f) \right).$$
Example: Consider \( x^4 + 9x + 9 \) and \( p = 3 \).
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**Example:** Consider \( x^4 + 9x + 9 \) and \( p = 3 \). We reduce modulo 3 and obtain \( x^4 \). The \( x \)-adic development is again \( x^4 + 9x + 9 \). Now the \( x \)-Newton polygon is:

![The x-Newton polygon](image)

The \( x \)-Newton polygon
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The Montes algorithm gives us a tool to deal with our small list of potentially problematic primes, but knowing the valuations of every coefficient of $\Psi_n$ is not easy.
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The Montes algorithm gives us a tool to deal with our small list of potentially problematic primes, but knowing the valuations of every coefficient of $\Psi_n$ is not easy.

However, if the constant coefficient has valuation 1 we don’t need to know about the other coefficients.
Stange has a paper where the valuations of the division polynomials evaluated at a point are explicitly computed. The valuations depend on the reduction data of the elliptic curve.
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We want to plug any point $P \in E(\overline{\mathbb{Q}})$ with $x(P) = 0$ into $\Psi_n$ so we can find the valuation of the constant coefficient.
Tate’s Algorithm

Given a curve

$$E : y^2 + (\alpha + 8\beta)xy + \beta(\alpha + 8\beta)^2y = x^3 + \beta(\alpha + 8\beta)x^2$$

in Tate’s normal form we apply Tate’s algorithm to understand the reduction type of $E$ in terms of $\alpha$ and $\beta$. 
Finally, we change coordinates to get to the Fueter form of the curve. The change of coordinates is...
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\[
(x, y) = \left( \frac{a\beta}{T} - a\beta, \frac{1}{2} \left( \frac{(a\beta)^{\frac{3}{2}} T_1}{T^2} - \frac{a^2 \beta}{T} \right) \right).
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Finally, we change coordinates to get to the Fueter form of the curve. The change of coordinates is...

\[(x, y) = \left( \frac{a\beta}{T} - a\beta, \frac{1}{2} \left( \frac{(a\beta)^{3/2} T_1}{T^2} - \frac{a^2 \beta}{T} \right) \right).\]

We apply the Montes algorithm to obtain the result. We’ve also shown that the odd Fueter division polynomials don’t yield monogenic fields for \( n > 3 \).
Further Questions
Can we use the Montes algorithm and explicit formulas for the discriminant of a polynomial to find other monogenic families?
Can an in-depth analysis of division polynomials, perhaps in conjunction with the Montes algorithm, shed light on some properties of torsion fields and torsion point fields?
Thank you for listening. Preprints of this work and some of the other work mentioned is available on my website:

http://math.colorado.edu/~hwsmith/index.html