# Local Index Theory over Foliation Groupoids

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**Abstract**

We give a local proof of an index theorem for a Dirac-type operator that is invariant with respect to the action of a foliation groupoid $G$. If $M$ denotes the space of units of $G$ then the input is a $G$-equivariant fiber bundle $P \to M$ along with a $G$-invariant fiberwise Dirac-type operator $D$ on $P$. The index theorem is a formula for the pairing of the index of $D$, as an element of a certain K-theory group, with a closed graded trace on a certain noncommutative de Rham algebra $\Omega^* \mathcal{B}$ associated to $G$. The proof is by means of superconnections in the framework of noncommutative geometry.

## 1 Introduction

It has been clear for some time, especially since the work of Connes [9] and Renault [27], that many interesting spaces in noncommutative geometry arise from groupoids. For background information, we refer to Connes' book [11, Chapter II]. In particular, to a smooth groupoid $G$ one can assign its convolution algebra $C_c^\infty(G)$, which represents a class of smooth functions on the noncommutative space specified by $G$.

An important motivation for noncommutative geometry comes from index theory. The notion of groupoid allows one to unify various index theorems that arise in the literature, such as the Atiyah-Singer families index theorem [2], the Connes-Skandalis foliation index

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theorem [13] and the Connes-Moscovici covering space index theorem [12]. All of these theorems can be placed in the setting of a proper cocompact action of a smooth groupoid $G$ on a manifold $P$. Given a $G$-invariant Dirac-type operator $D$ on $P$, the construction of [12] allows one to form its analytic index $\mathrm{Ind}_a$ as an element of the K-theory of the algebra $C_c^\infty(G) \otimes \mathcal{R}$, where $\mathcal{R}$ is an algebra of infinite matrices whose entries decay rapidly [11, Sections III.4, III.7.$\gamma$]. When composed with the trace on $\mathcal{R}$, the Chern character $\mathrm{ch}(\mathrm{Ind}_a)$ lies in the periodic cyclic homology group $\mathrm{PHC}_*(C_c^\infty(G))$. The index theorem, at the level of Chern characters, equates $\mathrm{ch}(\mathrm{Ind}_a)$ with a topological expression $\mathrm{ch}(\mathrm{Ind}_t)$.

We remark that in the literature, one often sees the analytic index defined as an element of K-theory of the groupoid $C^*$-algebra $C_r^*(G)$. The index in $\mathrm{K}_*(C_c^\infty(G) \otimes \mathcal{R})$ is a more refined object. However, to obtain geometric and topological consequences from the index theorem, it appears that one has to pass to $C_r^*(G)$; we refer to [11, Chapter III] for discussion. In this paper we will work with $C_c^\infty(G)$.

We prove a local index theorem for a Dirac-type operator that is invariant with respect to the action of a foliation groupoid. In the terminology of Crainic-Moerdijk [15], a foliation groupoid is a smooth groupoid $G$ with discrete isotropy groups, or equivalently, which is Morita equivalent to a smooth étale groupoid.

A motivation for our work comes from the Connes-Skandalis index theorem for a compact foliated manifold $(M, \mathcal{F})$ with a longitudinal Dirac-type operator [13]. To a foliated manifold $(M, \mathcal{F})$ one can associate its holonomy groupoid $G_{hol}$, which is an example of a foliation groupoid. The general foliation index theorem equates $\mathrm{Ind}_a$ with a topological index $\mathrm{Ind}_t$. For details, we refer to [11, Sections I.5, II.8-9, III.6-7].

We now state the index theorem that we prove. Let $M$ be the space of units of a foliation groupoid $G$. It carries a foliation $\mathcal{F}$. Let $\rho$ be a closed holonomy-invariant transverse current on $M$. There is a corresponding universal class $\omega_\rho \in \mathrm{H}^*(BG; o)$, where $o$ is a certain orientation character on the classifying space $BG$. Suppose that $G$ acts freely, properly and cocompactly on a manifold $P$. In particular, there is a submersion $\pi : P \to M$. There is an induced foliation $\pi^*\mathcal{F}$ of $P$ with the same codimension as $\mathcal{F}$, satisfying $T\pi^*\mathcal{F} = (d\pi)^{-1}T\mathcal{F}$. Let $g^{TZ}$ be a smooth $G$-invariant vertical Riemannian metric on $P$. Suppose that the vertical tangent bundle $TZ$ is even-dimensional and has a $G$-invariant spin structure. Let $S^Z$ be the corresponding vertical spinor bundle. Let $\widetilde{V}$ be an auxiliary $G$-invariant Hermitian vector bundle on $P$ with a $G$-invariant Hermitian connection. Put $E = S^Z \widehat{\otimes} \widetilde{V}$, a $G$-invariant $\mathbb{Z}_2$-graded Clifford bundle on $P$ which has a $G$-invariant connection. The Dirac-type operator $Q$ acts fiberwise on sections of $E$. Let $D$ be its restriction to the sections of positive parity. (The case of general $G$-invariant Clifford bundles $E$ is completely analogous.) Let $\mu : P \to P/G$ be the quotient map. Then $P/G$ is a smooth compact manifold with a foliation $F = (\pi^*\mathcal{F})/G$ satisfying $(d\mu)^{-1}TF = T\pi^*\mathcal{F}$. Put $V = \widetilde{V}/G$, a Hermitian vector bundle on $P/G$ with a Hermitian connection $\nabla^V$. The $G$-action on $P$ is classified by a map $\nu : P/G \to BG$, defined up to homotopy.

The main point of this paper is to give a local proof of the following theorem.

**Theorem 1**

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle \;=\; \int_{P/G} \widehat{A}(TF)\;\mathrm{ch}(V)\,\nu^*\omega_\rho. \tag{1}$$

Here $\mathrm{Ind}\, D$ lies in $\mathrm{K}_*(C_c^\infty(G) \otimes \mathcal{R})$. If $M$ is a compact foliated manifold and one takes $P = G = G_{hol}$ then one recovers the result of pairing the Connes-Skandalis theorem with $\rho$; see also Nistor [24].

In saying that we give a local proof of Theorem 1, the word "local" is in the sense of Bismut's proof of the Atiyah-Singer family index theorem [6]. In our previous paper [16] we gave a local proof of such a theorem in the étale case. One can reduce Theorem 1 to the étale case by choosing a complete transversal $T$, i.e. a submanifold of $M$, possibly disconnected, with $\dim(T) = \mathrm{codim}(\mathcal{F})$ and which intersects each leaf of the foliation. Using $T$, one can reduce the holonomy groupoid $G$ to a Morita-equivalent étale groupoid $G_{et}$. We gave a local proof of Connes' index theorem concerning an étale groupoid $G_{et}$ acting freely, properly and cocompactly on a manifold $P$, preserving a fiberwise Dirac-type operator $Q$ on $P$. Our local proof has since been used by Leichtnam and Piazza to prove an index theorem for foliated manifolds-with-boundary [21].

In the present paper we give a local proof of Theorem 1 working directly with foliation groupoids. In particular, the new proof avoids the noncanonical choice of a complete transversal $T$.

The overall method of proof is by means of superconnections in the context of noncommutative geometry, as in [16]. However, there are conceptual differences with respect to [16]. As in [16], we first establish an appropriate differential calculus on the noncommutative space determined by a foliation groupoid $G$. The notion of "smooth functions" on the noncommutative space is clear, and is given by the elements of the convolution algebra $\mathcal{B} = C_c^\infty(G)$. We define a certain graded algebra $\Omega^*\mathcal{B}$ which plays the role of the differential forms on the noncommutative space. The algebra $\Omega^*\mathcal{B}$ is equipped with a degree-1 derivation $d$, which is the analog of the de Rham differential. Unlike in the étale case, it turns out that in general, $d^2 \neq 0$. The reason for this is that to define $d$, we must choose a horizontal distribution $T^H M$ on $M$, where "horizontal" means transverse to $\mathcal{F}$. In general $T^H M$ is not integrable, which leads to the nonvanishing of $d^2$. This issue does not arise in the étale case.

As we wish to deal with superconnections in such a context, we must first understand how to do Chern-Weil theory when $d^2 \neq 0$. If $d^2$ is given by commutation with a 2-form then a trick of Connes [11, Chapter III.3, Lemma 9] allows one to construct a new complex with $d^2 = 0$, thereby reducing to the usual case. We give a somewhat more general formalism that may be useful in other contexts. It assumes that for the relevant $\mathcal{B}$-module $\mathcal{E}$ and connection $\nabla : \mathcal{E} \to \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$, there is a linear map $l : \mathcal{E} \to \Omega^2\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$ such that

$$l(b\xi) - \,b\,l(\xi) \;=\; d^2(b)\,\xi \tag{2}$$

and

$$l(\nabla\xi) \;=\; \nabla l(\xi) \tag{3}$$

for $b \in \mathcal{B}$, $\xi \in \mathcal{E}$. With this additional structure, we show in Section 2 how to do Chern-Weil theory, both for connections and superconnections on a $\mathcal{B}$-module $\mathcal{E}$. In the case when $d^2$ is a commutator, one recovers Connes' construction of Chern classes.

Next, we consider certain "homology classes" of the noncommutative space. A graded trace on $\Omega^*\mathcal{B}$ is said to be closed if it annihilates $\mathrm{Im}(d)$. A closed holonomy-invariant transverse current $\rho$ on the space of units $M$ gives a closed graded trace on $\Omega^*\mathcal{B}$.

The action of $G$ on $P$ gives rise to a left $\mathcal{B}$-module $\mathcal{E}$, which essentially consists of compactly-supported sections of $E$ coupled to a vertical density. We extend $\mathcal{E}$ to a left-$\Omega^*\mathcal{B}$ module $\Omega^*\mathcal{E}$ of "$\mathcal{E}$-valued differential forms". There is a natural linear map $l : \mathcal{E} \to \Omega^2\mathcal{E}$ satisfying (2) and (3).

We then consider the Bismut superconnection $A_s$ on $\mathcal{E}$. The formal expression for its Chern character involves $e^{-A_s^2 + l}$. The latter is well-defined in $\mathrm{Hom}^\omega(\mathcal{E}, \Omega^*\mathcal{E})$, an algebra consisting of rapid-decay kernels. We construct a graded trace $\tau : \mathrm{Hom}^\omega(\mathcal{E}, \Omega^*\mathcal{E}) \to \Omega^*\mathcal{B}$. This allows us to define the Chern character of the superconnection by

$$\mathrm{ch}(A_s) \ = \ \mathcal{R}\left(\tau e^{-A_s^2 + l}\right). \tag{4}$$

Here $\mathcal{R}$ is the rescaling operator which, for $p$ even, multiplies a $p$-form by $(2\pi i)^{-\frac{p}{2}}$.

Now let $\rho$ be a closed holonomy-invariant transverse current on $M$ as above. Then $\rho(\mathrm{ch}(A_s))$ is defined and we compute its limit when $s \to 0$, to obtain a differential form version of the right-hand-side of (1). (In the case when $P = G = G_{hol}$ an analogous computation was done by Heitsch [18, Theorem 2.1]).

Next, we use the argument of [16, Section 5] to show that for all $s > 0$, $\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle = \rho(\mathrm{ch}(A_s))$. (In the case when $P = G = G_{hol}$, this was shown under some further restrictions by Heitsch [18, Theorem 4.6] and Heitsch-Lazarov [19, Theorem 5].) This proves Theorem 1.

We note that our extension of [16] from étale groupoids to foliation groupoids is only partial. The local index theorem of [16] allows for pairing with more general objects than transverse currents, such as the Godbillon-Vey class. The paper [16] used a bicomplex $\Omega^{*,*}\mathcal{B}$ of forms, in which the second component consists of forms in the "noncommutative" direction. There was also a connection $\nabla$ on $\mathcal{E}$ which involved a differentiation in the noncommutative direction. In the setting of a foliation groupoid, one again has a bicomplex $\Omega^{*,*}\mathcal{B}$ and a connection $\nabla$. However, (3) is not satisfied. Because of this we work instead with the smaller complex of forms $\Omega^{*,0}\mathcal{B}$, where this problem does not arise.

The paper is organized as follows. In Section 2 we discuss Chern-Weil theory in the context of a graded algebra with derivation whose square is nonzero. In Section 3 we describe the differential algebra $\Omega^*\mathcal{B}$ associated to a foliation groupoid $G$. In Section 4 we add a manifold $P$ on which $G$ acts properly. We define a certain left-$\mathcal{B}$ module $\mathcal{E}$ and superconnection $A_s$ on

$\mathcal{E}$. We compute the $s \to 0$ limit of $\rho(\mathrm{ch}(A_s))$. In Section 5 we explain the relation between the superconnection computations and the K-theoretic index, construct the cohomology class $\omega_\rho \in \mathrm{H}^*(BG; o)$ and prove Theorem 1. We show that Theorem 1 implies some well-known index theorems.

In an appendix to this paper we give a technical improvement to our previous paper [16]. The index theorem in [16] assumed that the closed graded trace $\eta$ on $\Omega^*(B, \mathbb{C}\Gamma)$ extended to an algebra of rapidly decaying forms $\Omega^*(B, \mathcal{B}^\omega)$. The appearance of $\Omega^*(B, \mathcal{B}^\omega)$ was due to the noncompact support of the heat kernel, which affects the trace of the superconnection Chern character. In the appendix we show how to replace $\Omega^*(B, \mathcal{B}^\omega)$ by $\Omega^*(B, \mathbb{C}\Gamma)$, by using finite propagation speed methods. Let $f \in C_c^\infty(\mathbb{R})$ be a smooth even function with support in $[-\epsilon, \epsilon]$. Let $\widehat{f}$ be its Fourier transform. We can define $\widehat{f}(A_s)$ and show that $\eta\left(\mathcal{R}\,\tau \widehat{f}(A_s)\right)$ is defined for graded traces $\eta$ on $\Omega^*(B, \mathbb{C}\Gamma)$. We prove the corresponding analog of [16, Theorem 3], with the Gaussian function in the definition of the Chern character replaced by an appropriate function $\widehat{f}$. This then implies the result stated in [16, Theorem 3] without the condition of $\eta$ being extendible to $\Omega^*(B, \mathcal{B}^\omega)$. We remark that this issue of replacing $\Omega^*(B, \mathcal{B}^\omega)$ by $\Omega^*(B, \mathbb{C}\Gamma)$ does not arise in the present paper.

More detailed summaries are given at the beginnings of the sections.

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## 2 The Chern Character

In this section we collect some algebraic facts needed to define the Chern character of a superconnection in our setting. We consider an algebra $\mathcal{B}$ and a graded algebra $\Omega^*$ with $\Omega^0 = \mathcal{B}$. We assume that $\Omega^*$ is equipped with a degree-1 derivation $d$ whose square may be nonzero. If $\mathcal{E}$ is a left $\mathcal{B}$-module then the notion of a connection $\nabla$ on $\mathcal{E}$ is the usual one from noncommutative geometry; see Connes [11, Section III.3, Definition 5] and Karoubi [20, Chapitre 1]. We assume the additional structure of a map $l$ satisfying (2) and (3). We show that $\nabla^2 - l$ is then the right notion of curvature. If $\mathcal{E}$ is a finitely-generated projective $\mathcal{B}$-module then we carry out Chern-Weil theory for the connection $\nabla$, and show how it extends to the case of a superconnection $A$. Many of the lemmas in this section are standard in the case when $d^2 = 0$ and $l = 0$, but we present them in detail in order to make clear what goes through to the case when $d^2 \neq 0$. In the case when $d^2$ is given by a commutator, the Chern character turns out to be the same as what one would get using Connes' $X$-trick [11, Section III.3, Lemma 9].

Let $\mathcal{B}$ be an algebra over $\mathbb{C}$, possibly nonunital. Let $\Omega = \bigoplus_{i=1}^\infty \Omega^i$ be a graded algebra with $\Omega^0 = \mathcal{B}$. Let $d : \Omega^* \to \Omega^{*+1}$ be a graded derivation of $\Omega^*$. Define $\alpha : \Omega^* \to \Omega^{*+2}$ by $\alpha = d^2$;

then for all $\omega, \omega' \in \Omega^*$,

$$\alpha(d\omega) \;=\; d\alpha(\omega), \qquad \alpha(\omega\omega') \;=\; \alpha(\omega)\,\omega' \;+\; \omega\,\alpha(\omega'). \tag{5}$$

By a graded trace, we will mean a linear functional $\eta \,:\, \Omega^* \to \mathbb{C}$ such that

$$\eta(\alpha(\omega)) \;=\; 0, \qquad\qquad \eta([\omega, \omega']) \;=\; 0 \tag{6}$$

for all $\omega, \omega' \in \Omega^*$. Define $d^t\eta$ by $(d^t\eta)(\omega) \;=\; \eta(d\omega)$. Then the graded traces on $\Omega^*$ form a complex with differential $d^t$. A graded trace $\eta$ will be said to be closed if $d^t\eta \;=\; 0$, i.e. for all $\omega \in \Omega^*$, $\eta(d\omega) \;=\; 0$.

**Example 1 :** Let $E$ be a complex vector bundle over a smooth manifold $M$. Let $\nabla^E$ be a connection on $E$, with curvature $\theta^E \in \Omega^2(M; \mathrm{End}(E))$. Put $\mathcal{B} \,=\, C^\infty(M; \mathrm{End}(E))$ and $\Omega^* \;=\; \Omega^*(M; \mathrm{End}(E))$. Let $d$ be the extension of the connection $\nabla^E$ to $\Omega^*(M; \mathrm{End}(E))$. Then $\alpha(\omega) \;=\; \theta^E\,\omega \;-\; \omega\,\theta^E$. If $c$ is a closed current on $M$ then we obtain a closed graded trace $\eta$ on $\Omega^*$ by $\eta(\omega) \;=\; \int_c \mathrm{tr}(\omega)$.

Let $\mathcal{E}$ be a left $\mathcal{B}$-module. We assume that there is a $\mathbb{C}$-linear map $l \;:\; \mathcal{E} \to \Omega^2 \otimes_{\mathcal{B}} \mathcal{E}$ such that for all $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$,

$$l(b\xi) \;=\; \alpha(b)\,\xi \;+\; b\,l(\xi). \tag{7}$$

**Example 2 :** Suppose that for some $\theta \in \Omega^2$, $\alpha(\omega) \,=\, \theta\omega - \omega\theta$. Then we can take $l(\xi) \,=\, \theta\xi$.

**Lemma 1** *There is an extension of $l$ to a linear map $l \,:\, \Omega^* \otimes_{\mathcal{B}} \mathcal{E} \to \Omega^{*+2} \otimes_{\mathcal{B}} \mathcal{E}$ so that for $\omega \in \Omega^*$ and $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,*

$$l(\omega\mu) \;=\; \alpha(\omega)\,\mu \;+\; \omega\,l(\mu). \tag{8}$$

**PROOF.** We define $l \;:\; \Omega^* \otimes_{\mathbb{C}} \mathcal{E} \to \Omega^{*+2} \otimes_{\mathcal{B}} \mathcal{E}$ by

$$l(\omega \otimes \xi) \;=\; \alpha(\omega)\,\xi \;+\; \omega\,l(\xi). \tag{9}$$

Then for $b \in \mathcal{B}$,

$$\begin{aligned} l(\omega b \otimes \xi) \;&=\; \alpha(\omega b)\,\xi \;+\; \omega b\,l(\xi) \;=\; \alpha(\omega)\,b\,\xi \;+\; \omega\,\alpha(b)\,\xi \;+\; \omega\,b\,l(\xi) \\ &=\; \alpha(\omega)\,b\xi \;+\; \omega\,l(b\xi) \;=\; l(\omega \otimes b\xi). \end{aligned} \tag{10}$$

Thus $l$ is defined on $\Omega^* \otimes_{\mathcal{B}} \mathcal{E}$. Next, for $\omega, \omega' \in \Omega^*$ and $\xi \in \mathcal{E}$,

$$\begin{aligned} l(\omega(\omega'\xi)) \;&=\; \alpha(\omega\omega')\,\xi \;+\; \omega\omega'\,l(\xi) \;=\; \alpha(\omega)\,\omega'\,\xi \;+\; \omega\,\alpha(\omega')\,\xi \;+\; \omega\omega'\,l(\xi) \\ &=\; \alpha(\omega)\,\omega'\,\xi \;+\; \omega\,l(\omega'\,\xi). \end{aligned} \tag{11}$$

This proves the lemma.

Let $\nabla : \mathcal{E} \to \Omega^1 \otimes_{\mathcal{B}} \mathcal{E}$ be a connection, i.e. a $\mathbb{C}$-linear map satisfying

$$\nabla(b\xi) \;=\; db \otimes \xi \;+\; b\nabla\xi \tag{12}$$

for all $b \in \mathcal{B}$, $\xi \in \mathcal{E}$. Extend $\nabla$ to a $\mathbb{C}$-linear map $\nabla : \Omega^* \otimes_{\mathcal{B}} \mathcal{E} \to \Omega^{*+1} \otimes_{\mathcal{B}} \mathcal{E}$ so that for all $\omega \in \Omega^*$ and $\xi \in \mathcal{E}$,

$$\nabla(\omega\xi) \;=\; d\omega \otimes \xi \;+\; (-1)^{|\omega|}\,\omega\nabla\xi. \tag{13}$$

We assume that for all $\xi \in \mathcal{E}$,

$$l(\nabla\xi) \;=\; \nabla l(\xi). \tag{14}$$

**Lemma 2** *$\nabla^2 - l : \mathcal{E} \to \Omega^2 \otimes_{\mathcal{B}} \mathcal{E}$ is left-$\mathcal{B}$-linear.*

**PROOF.** For $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$,

$$\begin{aligned}(\nabla^2 - l)(b\xi) \;&=\; \nabla(db \otimes \xi \;+\; b\nabla\xi) \;-\; l(b\xi) \;=\; d^2 b \otimes \xi \;+\; b\nabla^2\xi \;-\; l(b\xi) \\ &=\; \alpha(b)\xi \;+\; b\nabla^2\xi \;-\; l(b\xi) \;=\; b\,(\nabla^2 - l)(\xi).\end{aligned} \tag{15}$$

This proves the lemma.

Put $\Omega^*_{ab} = \Omega^*/[\Omega^*, \Omega^*]$, the quotient by the graded commutator, with the induced $d$. For simplicity, in the rest of this section we assume that $\mathcal{B}$ is unital and $\mathcal{E}$ is a finitely-generated projective left $\mathcal{B}$-module. Consider the graded algebra $\mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E}) \;\cong\; \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$. There is a graded trace on $\mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E})$, with value in $\Omega^*_{ab}$, defined as follows. Write $\mathcal{E}$ as $\mathcal{B}^N e$ for some idempotent $e \in M_N(\mathcal{B})$. Then any $T \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E})$ can be represented as right-multiplication on $\mathcal{B}^N e$ by a matrix $T \in M_N(\Omega^*)$ satisfying $T = eT = Te$. By definition $\mathrm{tr}(T) = \sum_{i=1}^N T_{ii} \mod [\Omega^*, \Omega^*]$. It is independent of the representation of $\mathcal{E}$ as $\mathcal{B}^N e$.

Given $T_1, T_2 \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$, define their (graded) commutator by

$$[T_1, T_2] \;=\; T_1 \circ T_2 \;-\; (-1)^{|T_1||T_2|}\, T_2 \circ T_1. \tag{16}$$

For $T \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$, define $[\nabla, T] \in \mathrm{End}_{\mathbb{C}}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$ by

$$[\nabla, T](\mu) \;=\; (-1)^{|\mu|}\,(\nabla(T(\mu)) \;-\; T(\nabla\mu)) \tag{17}$$

for $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$.

**Lemma 3** $[\nabla, T] \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$.

**PROOF.** Given $\omega \in \Omega^*$ and $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,

$$\begin{aligned}
[\nabla, T](\omega\mu) &= (-1)^{|\omega|+|\mu|} \left(\nabla(T(\omega\mu)) \,-\, T(\nabla(\omega\mu))\right) \\
&= (-1)^{|\omega|+|\mu|} \left(\nabla(\omega T(\mu)) \,-\, T((d\omega)\mu \,+\, (-1)^{\omega}\omega\nabla\mu)\right) \\
&= (-1)^{|\omega|+|\mu|} \left((d\omega)T(\mu) \,+\, (-1)^{\omega}\omega\nabla(T(\mu)) \,-\, (d\omega)T(\mu) \,-\, (-1)^{\omega}\,\omega T(\nabla\mu)\right) \\
&= \omega\,[\nabla, T](\mu).
\end{aligned} \tag{18}$$

This proves the lemma.

**Lemma 4** *Given* $T_1, T_2 \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$,

$$[\nabla, T_1 \circ T_2] \;=\; T_1 \circ [\nabla, T_2] \;+\; (-1)^{|T_2|}\,[\nabla, T_1] \circ T_2. \tag{19}$$

**PROOF.** Given $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,

$$[\nabla, T_1 \circ T_2](\mu) \;=\; (-1)^{|\mu|}\left\{\nabla(T_1(T_2(\mu))) \,-\, T_1(T_2(\nabla(\mu)))\right\}, \tag{20}$$

$$\left(T_1 \circ [\nabla, T_2]\right)(\mu) \;=\; (-1)^{|\mu|}\,T_1\left(\nabla(T_2(\mu)) \,-\, T_2(\nabla(\mu))\right) \tag{21}$$

and

$$\left([\nabla, T_1] \circ T_2\right)(\mu) \;=\; [\nabla, T_1](T_2(\mu)) \;=\; (-1)^{|T_2(\mu)|}\left\{\nabla(T_1(T_2(\mu))) \,-\, T_1(\nabla(T_2(\mu)))\right\}. \tag{22}$$

The lemma follows.

**Lemma 5** *Given* $T_1, T_2 \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$,

$$[\nabla, [T_1, T_2]] \;=\; [T_1, [\nabla, T_2]] \;+\; (-1)^{|T_2|}\,[[\nabla, T_1], T_2]. \tag{23}$$

**PROOF.** This follows from (16) and (19). We omit the details.

**Lemma 6** *For* $T \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$,

$$\mathrm{tr}([\nabla, T]) \;=\; d\,\mathrm{tr}(T) \in \Omega^*_{ab}. \tag{24}$$

**PROOF.** Let us write $\mathcal{E} = \mathcal{B}^N e$ for an idempotent $e \in M_N(\mathcal{B})$. Given $A \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$, it acts on $\mathcal{B}^N e$ on the right by a matrix $A \in M_N(\Omega^1)$ with $A = eA = Ae$. Then there is some $A \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$ so that for $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E} = (\Omega^*)^N e$,

$$\nabla(\mu) \;=\; (d\mu)\,e \;+\; (-1)^{|\mu|}\,\mu\,A; \tag{25}$$

in fact, this equation defines $A$.

An element $T \in \mathrm{End}_{\Omega^*}(\Omega^* \otimes_{\mathcal{B}} \mathcal{E})$ acts by right multiplication on $\Omega^* \otimes_{\mathcal{B}} \mathcal{E} = (\Omega^*)^N e$ by a matrix $T \in M_N(\Omega^*)$ satisfying $T = eT = Te$. Then for $\xi \in \mathcal{E} = \mathcal{B}^N e$,

$$\begin{aligned}[\nabla, T](\xi) &= \nabla(\xi T) - (\nabla(\xi))T = \Big\{ d(\xi T)\, e + (-1)^{|T|}\, \xi T A \Big\} - \{(d\xi)\, e + \xi A\} T \qquad (26)\\ &= \xi \Big( (dT)\, e + (-1)^{|T|}\, TA - AT \Big)\end{aligned}$$

Thus $[\nabla, T]$ acts as right multiplication by the matrix

$$e(dT)e + (-1)^{|T|}\, TA - AT, \qquad (27)$$

and so $\mathrm{tr}([\nabla, T]) \equiv \mathrm{tr}(e(dT)e)$. On the other hand, using the identity $e(de)e = 0$ and taking the trace of $N \times N$ matrices, we obtain

$$\begin{aligned} d\,\mathrm{tr}(T) &= d\,\mathrm{tr}(eTe) = \mathrm{tr}\Big( (de)Te + e(dT)e + (-1)^{|T|}\, eT(de) \Big) \qquad (28)\\ &= \mathrm{tr}\Big( (de)eTe + e(dT)e + (-1)^{|T|}\, eTe(de) \Big) \\ &\equiv \mathrm{tr}\Big( e(de)eT + e(dT)e + (-1)^{|T|}\, Te(de)e \Big) = \mathrm{tr}(e(dT)e). \end{aligned}$$

This proves the lemma.

**Lemma 7** $[\nabla, \nabla^2 - l] = 0.$

**PROOF.** This follows from (14).

**Definition 1** *The Chern character form of* $\nabla$ *is*

$$\mathrm{ch}(\nabla) = \mathrm{tr}\left( e^{-\frac{\nabla^2 - l}{2\pi i}} \right) \in \Omega^*_{ab}. \qquad (29)$$

**Lemma 8** *Given* $\mathcal{E}$*, if* $\eta$ *is a closed graded trace on* $\Omega^*$ *then* $\eta(\mathrm{ch}(\nabla))$ *is independent of the choice of* $\nabla$*. If* $\eta_1$ *and* $\eta_2$ *are homologous closed graded traces then* $\eta_1(\mathrm{ch}(\nabla)) = \eta_2(\mathrm{ch}(\nabla))$.

**PROOF.** Let $\nabla_1$ and $\nabla_2$ be two connections on $\mathcal{E}$. For $t \in [0,1]$, define a connection by $\nabla(t) = t\nabla_2 + (1-t)\nabla_1$. Then $\frac{d\nabla}{dt} = \nabla_2 - \nabla_1 \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$. We claim that $\eta(\mathrm{ch}(\nabla(t)))$ is independent of $t$. As $\frac{d(\nabla^2 - l)}{dt} = \nabla \frac{d\nabla}{dt} + \frac{d\nabla}{dt}\nabla$, we have

$$\begin{aligned}\frac{d\,\mathrm{ch}(\nabla)}{dt} &= -\frac{1}{2\pi i}\,\mathrm{tr}\left( \left( \nabla \frac{d\nabla}{dt} + \frac{d\nabla}{dt}\nabla \right) e^{-\frac{\nabla^2 - l}{2\pi i}} \right) = -\frac{1}{2\pi i}\,\mathrm{tr}\left( \left[ \nabla, \frac{d\nabla}{dt}\, e^{-\frac{\nabla^2 - l}{2\pi i}} \right] \right) \qquad (30)\\ &= -\frac{1}{2\pi i}\, d\ \mathrm{tr}\left( \frac{d\nabla}{dt}\, e^{-\frac{\nabla^2 - l}{2\pi i}} \right).\end{aligned}$$

Then

$$\mathrm{ch}(\nabla_2) - \mathrm{ch}(\nabla_1) = -\frac{1}{2\pi i}\, d \int_0^1 \mathrm{tr}\left( (\nabla_2 - \nabla_1)\, e^{-\frac{\nabla(t)^2 - l}{2\pi i}} \right) dt, \qquad (31)$$

from which the claim follows. We note after expanding the exponential in (31), the integral gives an expression that is purely algebraic in $\nabla_1$ and $\nabla_2$.

If $\eta_1$ and $\eta_2$ are homologous then there is a graded trace $\eta'$ such that $\eta_1 - \eta_2 = d^t\eta'$. Thus

$$\eta_1(\mathrm{ch}(\nabla)) - \eta_2(\mathrm{ch}(\nabla)) = \eta'(d\,\mathrm{ch}(\nabla)). \tag{32}$$

However,

$$d\,\mathrm{ch}(\nabla) = d\,\mathrm{tr}\left(e^{-\frac{\nabla^2 - l}{2\pi i}}\right) = \mathrm{tr}\left(\left[\nabla, e^{-\frac{\nabla^2 - l}{2\pi i}}\right]\right) = 0. \tag{33}$$

This proves the lemma.

**Example 3 :** With the notation of Example 1, let $F$ be another complex vector bundle on $M$, with connection $\nabla^F$. Put $\mathcal{E} = C^\infty(M; E \otimes F)$, with $l(\xi) = (\theta^E \otimes I)\,\xi$ for $\xi \in \mathcal{E}$. Let $\nabla$ be the tensor product of $\nabla^E$ and $\nabla^F$. Then one finds that $\eta(\mathrm{ch}(\nabla)) = \int_c \mathrm{ch}(\nabla^F)$.

If $\mathcal{E}$ is $\mathbb{Z}_2$-graded, let $A : \mathcal{E} \to \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$ be a superconnection. Then there are obvious extensions of the results of this section. In particular, let $\mathcal{R}$ be the rescaling operator on $\Omega^{even}_{ab}$ which multiplies an element of $\Omega^{2k}_{ab}$ by $(2\pi i)^{-k}$.

**Definition 2** *The Chern character form of $A$ is*

$$\mathrm{ch}(A) = \mathcal{R}\ \mathrm{tr}_s\left(e^{-(A^2 - l)}\right) \in \Omega^*_{ab}. \tag{34}$$

We have the following analog of Lemma 8.

**Lemma 9** *Given $\mathcal{E}$, if $\eta$ is a closed graded trace on $\Omega^*$ then $\eta(\mathrm{ch}(A))$ is independent of the choice of $A$. If $\eta_1$ and $\eta_2$ are homologous closed graded traces then $\eta_1(\mathrm{ch}(A)) = \eta_2(\mathrm{ch}(A))$.*

## 3 Differential Calculus for Foliation Groupoids

In this section, given a foliation groupoid $G$, we construct a graded algebra $\Omega^*\mathcal{B}$ whose degree-0 component $\mathcal{B}$ is the convolution algebra of $G$. We then construct a degree-1 derivation $d = d^H$ of $\Omega^*\mathcal{B}$. Finally, we compute $d^2$.

### *3.1 The differential forms*

Let $G$ be a groupoid. We use the groupoid notation of [11, Section II.5]. The units of $G$ are denoted $G^{(0)}$ and the range and source maps are denoted $r, s : G \to G^{(0)}$. To construct the product of $g_0, g_1 \in G$, we must have $s(g_0) = r(g_1)$. Then $r(g_0 g_1) = r(g_0)$ and $s(g_0 g_1) = s(g_1)$. Given $m \in G^{(0)}$, put $G^m = r^{-1}(m)$, $G_m = s^{-1}(m)$ and $G^m_m = G^m \cap G_m$.

We assume that $G$ is a Lie groupoid, meaning that $G$ and $G^{(0)}$ are smooth manifolds, and $r$ and $s$ are smooth submersions. For simplicity we will assume that $G$ is Hausdorff. The results of the paper extend to the nonHausdorff case, using the notion of differential forms on a nonHausdorff manifold given by Crainic and Moerdijk [14, Section 2.2.5]. (The paper [14] is an extension of work by Brylinski and Nistor [8].)

The Lie algebroid $\mathfrak{g}$ of $G$ is a vector bundle over $G^{(0)}$ with fibers $\mathfrak{g}_m = T_m G^m = \mathrm{Ker}(dr_m : T_m G \to T_m G^{(0)})$. The anchor map $\mathfrak{g} \to TG^{(0)}$, a map of vector bundles, is the restriction of $ds_m : T_m G \to T_m G^{(0)}$ to $\mathfrak{g}_m$. In general, the image of the anchor map need not be of constant rank.

We now assume that $G$ is a foliation groupoid in the sense of [15], i.e. that $G$ satisfies one of the three following equivalent conditions [15, Theorem 1] :
1. $G$ is Morita equivalent to a smooth étale groupoid.
2. The anchor map of $G$ is injective.
3. All isotropy Lie groups $G^m_m$ of $G$ are discrete.

**Example 4 :** If $G$ is an smooth étale groupoid then $G$ is a foliation groupoid. If $(M, \mathcal{F})$ is a smooth foliated manifold then its holonomy groupoid (see Connes [11, Section II.8.$\alpha$]) and its monodromy (= homotopy) groupoid (see Baum-Connes [3] and Phillips [26]) are foliation groupoids. In this case, the anchor map is the inclusion map $T\mathcal{F} \to TM$. If a Lie group $L$ acts smoothly on a manifold $M$ and the isotropy groups $L_m = \{l \in L : ml = m\}$ are discrete then the cross-product groupoid $M \rtimes L$ is a foliation groupoid.

Put $M = G^{(0)}$. It inherits a foliation $\mathcal{F}$, with the leafwise tangent bundle $T\mathcal{F}$ being the image of the anchor map.

Note that the foliated manifold $(M, \mathcal{F})$ has a holonomy groupoid Hol which is itself a foliation groupoid. However, Hol may not be the same as $G$. If $G$ is a foliation groupoid with the property that $G_m$ is connected for all $m$ then $G$ lies between the holonomy groupoid of $\mathcal{F}$ and the monodromy groupoid of $\mathcal{F}$; see [15, Proposition 1] for further discussion. The reader may just want to keep in mind the case when $G$ is actually the holonomy groupoid of a foliated manifold $(M, \mathcal{F})$.

Let $\tau = TM/T\mathcal{F}$ be the normal bundle to the foliation. Given $g \in G$, let $U \subset M$ be a sufficiently small neighborhood of $s(g)$ and let $c : U \to G$ be a smooth map such that $c(s(g)) = g$ and $s \circ c = \mathrm{Id}_U$. Then $d(r \circ c)_{s(g)} : T_{s(g)} M \to T_{r(g)} M$ sends $T_{s(g)}\mathcal{F}$ to $T_{r(g)}\mathcal{F}$. The induced map from $\tau_{s(g)}$ to $\tau_{r(g)}$ has an inverse $g_* : \tau_{r(g)} \to \tau_{s(g)}$ called the holonomy of the element $g \in G$. It is independent of the choices of $U$ and $c$.

Let $\mathcal{D}$ denote the real line bundle on $M$ formed by leafwise densities. We define a graded algebra $\Omega^*\mathcal{B}$ whose components, as vector spaces, are given by

$$\Omega^n\mathcal{B} \;=\; C^\infty_c\left(G;\Lambda^n(r^*\tau^*)\otimes s^*\mathcal{D}\right) \tag{35}$$

In particular,

$$\mathcal{B} \;=\; \Omega^0\mathcal{B} \;=\; C^\infty_c(G;s^*\mathcal{D}) \tag{36}$$

is the groupoid algebra. (Instead of using half-densities, we have placed a full density at the source.) The product of $\phi_1 \in \Omega^{n_1}\mathcal{B}$ and $\phi_2 \in \Omega^{n_2}\mathcal{B}$ is given by

$$(\phi_1\phi_2)(g) \;=\; \int_{g'g''=g} \phi_1(g') \;\wedge\; \phi_2(g''). \tag{37}$$

In forming the wedge product, the holonomy of $g'$ is used to identify conormal spaces.

Let $T^HM$ be a horizontal distribution on $M$, i.e. a splitting of the short exact sequence $0 \to T\mathcal{F} \to TM \to \tau \to 0$. Then there is a horizontal differentiation $d^H : \Omega^n\mathcal{B} \to \Omega^{n+1}\mathcal{B}$, which we now define. The definition will proceed by building up $d^H$ from smaller pieces (compare [11, Section II.7.$\alpha$, Proposition 3]).

First, the choice of horizontal distribution allows us to define a horizontal differential $d^H$ : $\Omega^*(M) \to \Omega^{*+1}(M)$ as in Bismut-Lott [7, Definition 3.2] and Connes [11, Section III.7.$\alpha$]. Using the local description of an element of $C^\infty(M;\mathcal{D})$ as a vertical $\dim(\mathcal{F})$-form on $M$, we also obtain a horizontal differential $d^H : C^\infty(M;\mathcal{D}) \to C^\infty(M;\tau^*\otimes\mathcal{D})$ [11, Section III.7.$\alpha$] and a horizontal differential $d^H : C^\infty(M;\Lambda^n\tau^*) \to C^\infty(M;\Lambda^{n+1}\tau^*)$.

Given $f \in C^\infty_c(G)$, we now define its horizontal differential $d^Hf \in C^\infty_c(G;r^*\tau^*)$ by simultaneously differentiating $f$ with respect to its arguments, in a horizontal direction. That is, consider a point $g \in G$ and a vector $X_0 \in \tau_{r(g)}$. Put $X_1 = g_*(X_0)$. Next, use the horizontal distribution $T^HM$ to construct the corresponding horizontal vectors $\widetilde{X_0}$ and $\widetilde{X_1}$. We now have a vector $\widetilde{X} = \left(\widetilde{X_0},\widetilde{X_1}\right) \in T_{(r(g),s(g))}(M\times M)$. It is the image of a unique vector $X \in T_gG$ under the immersion

$$(r,s) \;:\; G \to M\times M. \tag{38}$$

We define $d^Hf$ by putting $\left((d^Hf)(X_0)\right)(g) \;=\; Xf$.

Next, to horizontally differentiate an element of $C^\infty_c\left(G;\Lambda^n(r^*\tau^*)\otimes s^*\mathcal{D}\right)$, we write it as a finite sum of terms of the form $f\,r^*(\omega)\,s^*(\beta)$, with $f \in C^\infty_c(G)$, $\omega \in C^\infty(M;\Lambda^n\tau^*)$, and $\beta \in C^\infty(M;\mathcal{D})$. For an element of this form, put

$$d^H\left(f\,r^*(\omega)\,s^*(\beta)\right) \;=\; (d^Hf)\,r^*(\omega)\,s^*(\beta) \;+\; f\,r^*(d^H\omega)\,s^*(\beta) \;+\; (-1)^n\,f\,r^*(\omega)\,s^*(d^H\beta), \tag{39}$$

where the holonomy is used in defining products.

**Lemma 10** *The operator $d^H$ is a graded derivation of $\Omega^*\mathcal{B}$.*

**PROOF.** This follows from a straightforward computation, which we omit.

Put $d = d^H$. We now describe $\alpha = d^2$. Let $T \in \Omega^2(M; T\mathcal{F})$ be the curvature of the horizontal distribution $T^H M$ [7, (3.11)]. It is a horizontal 2-form on $M$ with values in $T\mathcal{F}$, defined by $T(X_1, X_2) = -P^{vert}[X_1^H, X_2^H]$. One can define the Lie derivative $\mathcal{L}_T : \Omega^*(M) \to \Omega^{*+2}(M)$, an operation which increases the horizontal grading by two, as in [7, (3.14)]. Then one can define $\mathcal{L}_T : C^\infty(M; \mathcal{D}) \to C^\infty(M; \Lambda^2 \tau^* \otimes \mathcal{D})$ and $\mathcal{L}_T : C^\infty(M; \Lambda^n \tau^*) \to C^\infty(M; \Lambda^{n+2} \tau^*)$ in obvious ways.

Given $f \in C_c^\infty(G)$, we define its Lie derivative $\mathcal{L}_T f \in C_c^\infty(G; \Lambda^2(r^* \tau^*))$ by simultaneously differentiating $f$ with respect to its arguments, in the vertical direction. That is, consider a point $g \in G$ and $X_0, Y_0 \in \tau_{r(g)}$. Put $X_1 = g_*(X_0)$ and $Y_1 = g_*(Y_0)$. Next, use the horizontal distribution $T^H M$ to construct the corresponding horizontal vectors $\widetilde{X_0}$, $\widetilde{X_1}$, $\widetilde{Y}_0$ and $\widetilde{Y}_1$. Consider the vertical vectors $T(\widetilde{X_0}, \widetilde{Y}_0) \in T_{r(g)}\mathcal{F}$ and $T(\widetilde{X_1}, \widetilde{Y}_1) \in T_{s(g)}\mathcal{F}$. We now have a total vector $\widetilde{V} = \left( T(\widetilde{X_0}, \widetilde{Y}_0), T(\widetilde{X_1}, \widetilde{Y}_1) \right) \in T_{(r(g), s(g))}(M \times M)$. It is the image of a unique vector $V \in T_g G$ under the immersion (38). We define $\mathcal{L}_T f$ by putting $((\mathcal{L}_T f)(X_0, Y_0))(g) = Vf$.

Now for $f\, r^*(\omega)\, s^*(\beta)$ as before, we put

$$\mathcal{L}_T \left( f\, r^*(\omega)\, s^*(\beta) \right) = (\mathcal{L}_T f)\, r^*(\omega)\, s^*(\beta) + f\, r^*(\mathcal{L}_T \omega)\, s^*(\beta) + f\, r^*(\omega)\, s^*(\mathcal{L}_T \alpha_1), \quad (40)$$

where the holonomy is used in defining products.

**Lemma 11** *We have*

$$\alpha = -\mathcal{L}_T. \quad (41)$$

**PROOF.** This follows from the method of proof of [7, (3.13)] or [11, Section III.7.$\alpha$].

**Remark :** One can consider $\alpha$ to be commutation with a (distributional) element of the multiplier algebra $C^{-\infty}(G; \Lambda^2(p_0^* \tau^*) \otimes p_1^* \mathcal{D})$, namely the one that implements the Lie differentiation [11, Section III.7.$\alpha$, Lemma 4].

## 4 Superconnection and Chern character

In this section we consider a smooth manifold $P$ on which $G$ acts freely, properly and cocompactly, along with a $G$-invariant $\mathbb{Z}_2$-graded vector bundle $E$ on $P$. We construct a corresponding left-$\mathcal{B}$-module $\mathcal{E}$. Given a $G$-invariant Dirac-type operator which acts on sections of $E$, we consider the Bismut superconnections $\{A_s\}_{s>0}$. We compute the $s \to 0$ limit of the pairing between the Chern character of $A_s$ and a closed graded trace on $\Omega^* \mathcal{B}$

that is concentrated on the units $M$. More detailed summaries appear at the beginnings of the subsections.

### *4.1 Module and Connection*

In this subsection we consider a left $\mathcal{B}$-module $\mathcal{E}$ consisting of sections of $E$, and its extension to a left $\Omega^*\mathcal{B}$-module $\Omega^*\mathcal{E}$. We construct a map $l : \mathcal{E} \to \Omega^2\mathcal{E}$ satisfying (2). Given a lift $T^H P$ of $T^H M$, we construct a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.

Let $P$ be a smooth $G$-manifold [11, Section II.10.$\alpha$, Definition 1]. That is, first of all, there is a submersion $\pi : P \to M$. Given $m \in M$, we write $Z_m = \pi^{-1}(m)$. Putting

$$P \times_r G = \{(p, g) \in P \times G : p \in Z_{r(g)}\}, \tag{42}$$

we must also have a smooth map $P \times_r G \to P$, denoted $(p, g) \to pg$, such that $pg \in Z_{s(g)}$ and $(pg_1)g_2 = p(g_1g_2)$ for all $(g_1, g_2) \in G^{(2)}$. It follows that for each $g \in G$, the map $p \to pg$ gives a diffeomorphism from $Z_{r(g)}$ to $Z_{s(g)}$. Let $\mathcal{D}_Z$ denote the real line bundle on $P$ formed by the fiberwise densities.

Hereafter we assume that $P$ is a proper $G$-manifold [11, Section II.10.$\alpha$, Definition 2], i.e. that the map $P \times_r G \to P \times P$ given by $(p, g) \to (p, pg)$ is proper. We also assume that $G$ acts cocompactly on $P$, i.e. that the quotient of $P$ by the equivalence relation ($p \sim p'$ if $p = p'g$ for some $g \in G$) is compact. And we assume that $G$ acts freely on $P$, i.e. that $pg = p$ implies that $g \in M$. Then $P/G$ is a smooth compact manifold.

**Example 5 :** Take $P = G$, with $\pi = s$. Then $G$ acts properly, freely, and, if $M$ is compact, cocompactly on $P$.

We will say that a covariant object (vector bundle, connection, metric, etc.) on $P$ is $G$-invariant if it is the pullback of a similar object from $P/G$. Let $E$ be a $G$-invariant $\mathbb{Z}_2$-graded vector bundle on $P$, with supertrace $\mathrm{tr}_s$ on $\mathrm{End}(E)$. Put $\mathcal{E} = C_c^\infty(P; E)$. It is a left-$\mathcal{B}$-module, with the action of $b \in \mathcal{B}$ on $\xi \in \mathcal{E}$ given by

$$(b\xi)(p) = \int_{G^{\pi(p)}} b(g)\, \xi(pg). \tag{43}$$

In writing (43), we have used the $g$-action to identify $E_p$ and $E_{pg}$.

Put

$$\Omega^n\mathcal{E} = C_c^\infty\left(P; \Lambda^n(\pi^*\tau^*) \otimes E\right). \tag{44}$$

Then $\Omega^*\mathcal{E}$ is a left-$\Omega^*\mathcal{B}$-module with the action of $\Omega^*\mathcal{B}$ on $\Omega^*\mathcal{E}$ given by

$$(\phi\, \omega)(p) = \int_{G^{\pi(p)}} \phi(g) \wedge \omega(pg). \tag{45}$$

Let $\widetilde{\mathcal{F}}$ be the foliation on $P$ whose leaf through $p \in P$ consists of the elements $pg$ where $g$ runs through the connected component of $G^{\pi(p)}$ that contains the unit $\pi(p)$. Note that $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$. Given $p \in P$ and $X, Y \in \tau_{\pi(p)}$, let $\widetilde{T}(X,Y) \in T_p\widetilde{\mathcal{F}}$ be the lift of $T(X,Y) \in T_{\pi(p)}\mathcal{F}$. Define $l : \mathcal{E} \to \Omega^2\mathcal{E}$ by saying that for $X, Y \in \tau_{\pi(p)}$ and $\xi \in \mathcal{E}$,

$$(l(\xi)(X,Y))(p) \;=\; -\,\widetilde{T}(X,Y)\xi. \tag{46}$$

Here we have used the $G$-invariance of $E$ to define the action of $\widetilde{T}(X,Y)$ on $\xi$.

**Lemma 12** *For all $X, Y \in \tau_{\pi(p)}$, $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$,*

$$l(b\xi) \;=\; \alpha(b)\,\xi \;+\; b\,l(\xi). \tag{47}$$

**PROOF.** We have

$$(l(b\xi)(X,Y))(p) \;=\; -\,\widetilde{T}(X,Y)\int_{G^{\pi(p)}} b(g)\,\xi(pg) \;=\; -\int_{G^{\pi(p)}} T(X,Y)b(g)\,\xi(pg), \tag{48}$$

$$(\alpha(b)(X,Y)\xi)(p) \;=\; -\int_{G^{\pi(p)}} \left(T(X,Y)b \;+\; T(g_*X, g_*Y)b\right)(g)\,\xi(pg) \tag{49}$$

and

$$(bl(\xi)(X,Y))(p) \;=\; -\int_{G^{\pi(p)}} b(g)\,\widetilde{T}(g_*X, g_*Y)\xi(pg). \tag{50}$$

Then

$$\begin{aligned}&(l(b\xi)(X,Y))(p) \;-\; (\alpha(b)(X,Y)\xi)(p) \;-\; (bl(\xi)(X,Y))(p) \;=\\ &\int_{G^{\pi(p)}} \Big(T(g_*X, g_*Y)b(g)\,\xi(pg) \;+\; b(g)\,\widetilde{T}(g_*X, g_*Y)\xi(pg)\Big)\,.\end{aligned} \tag{51}$$

We can write (51) more succinctly as

$$l(b\xi) \;-\; \alpha(b)\xi \;-\; bl(\xi) \;= \int_{G^{\pi(p)}} \mathcal{L}_{\widetilde{T}}(b(g)\xi(pg)), \tag{52}$$

where the Lie differentiation is at $pg$. The right-hand-side of (52) vanishes, being the integral of a Lie derivative of a compactly-supported density.

We extend $l$ to a linear map $l : \Omega^n\mathcal{E} \to \Omega^{n+2}\mathcal{E}$ as Lie differentiation in the $\widetilde{T}$-direction with respect to $P$.

**Lemma 13** *For all $\omega \in \Omega^*\mathcal{B}$ and $\mu \in \Omega^*\mathcal{E}$,*

$$l(\omega\mu) \;=\; \alpha(\omega)\,\mu \;+\; \omega\,l(\mu). \tag{53}$$

**PROOF.** The proof is similar to that of Lemma 12. We omit the details.

There is a pullback foliation $\pi^*\mathcal{F}$ on $P$ with the same codimension as $\mathcal{F}$, satisfying $T\pi^*\mathcal{F} = (d\pi)^{-1}T\mathcal{F}$. Let $\mu : P \to P/G$ be the quotient map. Then $P/G$ is a smooth compact manifold with a foliation $F = (\pi^*\mathcal{F})/G$ satisfying $(d\mu)^{-1}TF = T\pi^*\mathcal{F}$. We note that the normal bundle $NF$ to $F$ satisfies $\mu^*NF = \pi^*\tau$.

Let $T^H(P/G)$ be a horizontal distribution on $P/G$, transverse to $F$. Then $(d\mu)^{-1}(T^H(P/G))$ is a $G$-invariant distribution on $P$ that is transverse to the vertical tangent bundle $TZ$. Put $T^HP = (d\mu)^{-1}(T^H(P/G)) \cap (d\pi)^{-1}(T^HM)$, a distribution on $P$ that is transverse to $\pi^*\mathcal{F}$ and that projects isomorphically under $\pi$ to $T^HM$.

Let $\nabla^{\mathcal{E}} : \mathcal{E} \to \Omega^1\mathcal{E}$ be covariant differentiation on $\mathcal{E} = C_c^\infty(P; E)$ with respect to $T^HP$.

**Lemma 14** *$\nabla^{\mathcal{E}}$ is a connection.*

**PROOF.** We wish to show that

$$\nabla^{\mathcal{E}}(b\xi) = b\nabla^{\mathcal{E}}\xi + (d^Hb)\xi. \tag{54}$$

As the claim of the lemma is local on $P$, consider first the case when $T^H(P/G)$ is integrable. Let $T^HP_1$ and $\nabla_1^{\mathcal{E}}$ denote the corresponding objects on $P$. Then one is geometrically in a product situation and one can reduce to the case $P = M$, where one can check that (54) holds. If $T^H(P/G)$ is not integrable then $T^HP - T^HP_1 \in \mathrm{Hom}(\pi^*\tau, TZ)$ is the pullback under $\mu$ of an element of $\mathrm{Hom}(NF, TF)$. Hence $T^HP - T^HP_1$ is $G$-invariant and it follows that $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ commutes with $\mathcal{B}$, which proves the lemma.

We extend $\nabla^{\mathcal{E}}$ to act on $\Omega^*\mathcal{E}$ so as to satisfy Leibnitz' rule.

**Lemma 15** *For all $\xi \in \mathcal{E}$,*

$$l(\nabla^{\mathcal{E}}\xi) = \nabla^{\mathcal{E}}l(\xi). \tag{55}$$

**PROOF.** As $d^H$ commutes with $(d^H)^2$, it follows that $d^H$ commutes with $\mathcal{L}_T$. As the claim of the lemma is local on $P$, consider first the case when $T^H(P/G)$ is integrable. Let $T^HP_1$ and $\nabla_1^{\mathcal{E}}$ denote the corresponding objects on $P$. Then one is in a local product situation and the lemma follows from the fact that $d^H$ commutes with $\mathcal{L}_T$. If $T^H(P/G)$ is not integrable then $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ is given by covariant differentiation in the $TZ$ direction, with respect to $T^HP - T^HP_1 \in \mathrm{Hom}(\pi^*\tau, TZ)$. As $\widetilde{T}$ pulls back from $M$, $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ commutes with $l$. The lemma follows.

*4.2 Supertraces*

In this subsection we consider a certain algebra $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$ of operators with smooth kernel on $P$. We show that a trace on $\mathcal{B}$, concentrated on the units $M$, gives a supertrace on $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$. We then consider an algebra $\mathrm{Hom}^\infty_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$ of form-valued operators. We show that a closed graded trace on $\Omega^*\mathcal{B}$, concentrated on $M$, gives rise to a closed graded trace on $\mathrm{Hom}^\infty_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$.

An operator $K \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ has a Schwartz kernel $K(p'|p)$ so that

$$(K\xi)(p) \;=\; \int_{Z_{\pi(p)}} \xi(p')\, K(p'|p). \tag{56}$$

Define $q', q \;:\; P \times_M P \to P$ by $q'(p', p) \;=\; p'$ and $q(p', p) \;=\; p$. Let $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$ denote the subalgebra of $\mathrm{End}_{\mathcal{B}}(\mathcal{E})$ consisting of operators whose Schwartz kernel lies in $C^\infty_c(P \times_M P; (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E))$.

Choose $\Phi \in C^\infty_c(P; \pi^*\mathcal{D})$ so that

$$\int_{G^{\pi(p)}} \Phi(pg) = 1 \tag{57}$$

for all $p \in P$; that such a $\Phi$ exists was shown by Tu [30, Proposition 6.11]. Define $\tau K \in C^\infty_c(M; \mathcal{D})$ by

$$(\tau K)(m) \;=\; \int_{Z_m} \Phi(p)\ \mathrm{tr}_s\, K(p|p). \tag{58}$$

**Proposition 1** *Let $\rho$ be a linear functional on $C^\infty_c(M; \mathcal{D})$. Suppose that the linear functional $\eta$ on $\mathcal{B}$, defined by*

$$\eta(b) \;=\; \rho(b\big|_M), \tag{59}$$

*is a trace on $\mathcal{B}$. Then $\rho \circ \tau$ is a supertrace on* $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$.

**PROOF.** Consider the algebra $\mathrm{End}_{C^\infty_c(M)}(\mathcal{E})$. An operator $K \in \mathrm{End}_{C^\infty_c(M)}(\mathcal{E})$ has a Schwartz kernel $K(p|p')$ so that

$$(K\xi)(p) \;=\; \int_{Z_{\pi(p)}} K(p|p')\, \xi(p'). \tag{60}$$

(Note the difference in ordering as compared to (56).) For this proof, define $q, q' : P \times_M P \to P$ by $q(p, p') = p$ and $q'(p, p') = p'$. Let $\mathrm{End}^\infty_{C^\infty_c(M)}(\mathcal{E})$ denote the subalgebra of $\mathrm{End}_{C^\infty_c(M)}(\mathcal{E})$ consisting of operators whose Schwartz kernel lies in $C^\infty_c(P \times_M P; q^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E)$. The product in $\mathrm{End}^\infty_{C^\infty_c(M)}(\mathcal{E})$ is given by

$$(KK')(p|p') \;=\; \int_{p''} K(p|p'')\, K'(p''|p'). \tag{61}$$

Note that an element of $\mathrm{End}^\infty_{C^\infty_c(M)}(\mathcal{E})$ is not necessarily $G$-invariant. Note also that there is an injective homomorphism $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E}) \to \mathrm{End}^\infty_{C^\infty_c(M)}(\mathcal{E})^{op}$, where *op* denotes the opposite algebra, i.e. with the transpose multiplication. There is a fiberwise $G$-invariant supertrace $Tr_s : \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E}) \rightarrow C_c^\infty(M)$ given by

$$(Tr_s\, K)(m) \;=\; \int_{Z_m} \mathrm{tr}_s\, K(p|p). \tag{62}$$

Consider the algebra $\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$. The product in the algebra takes into account the action of $\mathcal{B}$ on $\mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$, which derives from the $G$-action on $P$. An element of the algebra has a kernel $K(g, p|p')$, where $p, p' \in Z_{s(g)}$. The product is given by

$$(K_1 K_2)(g, p|p') \;=\; \int_{g'g''=g} \int_{p'' \in Z_{s(g')}} K_1(g', p(g'')^{-1}|p'')\, K_2(g'', p''g''|p'). \tag{63}$$

The supertrace (62) induces a map $Tr_s : \mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E}) \rightarrow \mathcal{B}$ by

$$(Tr_s\, K)(g) \;=\; \int_{Z_{s(g)}} \mathrm{tr}_s\, K(g, p|p). \tag{64}$$

**Lemma 16** *$\eta \circ Tr_s$ is a supertrace on $\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$.*

**PROOF.** We can formally write

$$(\eta \circ Tr_s)(K) \;=\; \int_M \rho(m) \int_{Z_m} \mathrm{tr}_s\, K(m, p|p), \tag{65}$$

keeping in mind that $\rho$ is actually distributional. Then

$$\begin{aligned}
(\eta \circ Tr_s)(K_1K_2) \;&=\; \int_{g' \in G} \int_{p \in Z_{r(g')}} \int_{p'' \in Z_{s(g')}} \rho(r(g'))\ \mathrm{tr}_s \left( K_1(g', pg'|p'')\, K_2((g')^{-1}, p''(g')^{-1}|p) \right) \qquad (66)\\
&=\; \int_{g' \in G} \int_{p \in Z_{r(g')}} \int_{p'' \in Z_{s(g')}} \rho(r(g'))\ \mathrm{tr}_s \left( K_2((g')^{-1}, p''(g')^{-1}|p)\, K_1(g', pg'|p'') \right)\\
&=\; \int_{g' \in G} \int_{p'' \in Z_{r(g')}} \int_{p \in Z_{s(g')}} \rho(s(g'))\ \mathrm{tr}_s \left( K_2(g', p''g'|p)\, K_1((g')^{-1}, p(g')^{-1}|p'') \right).
\end{aligned}$$

However, the fact that $\eta$ is a trace on $\mathcal{B}$ translates into the fact that

$$\int_{g \in G} \rho(s(g))\, f(g) \;=\; \int_{g \in G} \rho(r(g))\, f(g) \tag{67}$$

for all $f \in C_c^\infty(G)$, from which the lemma follows.

We define a map $i : \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E}) \rightarrow \left( \mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E}) \right)^{op}$ by

$$(i(K))(g, p|p') \;=\; \Phi(pg^{-1}) K(p|p'). \tag{68}$$

**Lemma 17** *The map $i$ is a homomorphism.*

**PROOF.** Given $K_1, K_2 \in \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$, we have

$$\begin{aligned}
(i(K_1)\, i(K_2))\,(g, p|p') &= \int_{g'g''=g} \int_{Z_{s(g')}} i(K_1)(g', p(g'')^{-1}|p'')\, i(K_2)(g'', p''g''|p') \qquad (69)\\
&= \int_{g'g''=g} \int_{Z_{s(g')}} \Phi(pg^{-1})\, K_1(p(g'')^{-1}|p'')\, \Phi(p'')\, K_2(p''g''|p')\\
&= \int_{g'g''=g} \int_{Z_{s(g')}} \Phi(pg^{-1})\, K_1(p|p''g'')\, \Phi(p'')\, K_2(p''g''|p')\\
&= \int_{g'g''=g} \int_{Z_{s(g'')}} \Phi(pg^{-1})\, K_1(p|p'')\, \Phi(p''(g'')^{-1})\, K_2(p''|p')\\
&= \Phi(pg^{-1}) \int_{Z_{s(g)}} K_1(p|p'')\, K_2(p''|p')\\
&= (i(K_2K_1))(g, p|p').
\end{aligned}$$

Thus $i$ gives a homomorphism from $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})^{op}$ to $\mathcal{B} \otimes_{C^\infty_c(M)} \mathrm{End}^\infty_{C^\infty_c(M)}(\mathcal{E})$, from which the lemma follows.

**Lemma 18** *We have $\eta \circ Tr_s \circ i \;=\; \rho \circ \tau$.*

**PROOF.** Given $K \in \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$, we have

$$\begin{aligned}
(\eta \circ Tr_s \circ i)(K) &= \int_M \rho(m) \int_{Z_m} \mathrm{tr}_s(i(K))(m, p|p) \qquad (70)\\
&= \int_M \rho(m) \int_{Z_m} \Phi(p)\; \mathrm{tr}_s\, K(p|p) \;=\; (\rho \circ \tau)(K).
\end{aligned}$$

This proves the lemma.

Proposition 1 now follows from Lemmas 16-18.

**Example 6 :** Let $\mu$ be a holonomy-invariant transverse measure for $\mathcal{F}$. Let $\{U_i\}_{i=1}^N$ be an open covering of $M$ by flowboxes, with $U_i = V_i \times W_i$, $V_i \subset \mathbb{R}^{codim(\mathcal{F})}$ and $W_i \subset \mathbb{R}^{dim(\mathcal{F})}$. Let $\mu_i$ be the measure on $V_i$ which is the restriction of $\mu$. Let $\{\phi_i\}_{i=1}^N$ be a partition of unity that is subordinate to $\{U_i\}_{i=1}^N$. For $f \in C^\infty_c(M; \mathcal{D})$, put $\rho(f) = \sum_{i=1}^N \int_{V_i} \left( \int_{W_i} \phi_i\, f \right) d\mu_i$. Then $\rho$ satisfies the hypotheses of Proposition 1.

An operator $K \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$ has a Schwartz kernel $K(p'|p)$ so that

$$(K\xi)(p) = \int_{Z_{\pi(p)}} \xi(p')\, K(p'|p). \qquad (71)$$

Let $\mathrm{Hom}^\infty_{\mathcal{B}}(\mathcal{E}, \Omega^n \mathcal{E})$ denote the subspace of $\mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^n \mathcal{E})$ consisting of operators whose Schwartz kernel lies in

$$C^\infty_c(P \times_M P; \Lambda^n((\pi \circ q)^* \tau^*) \otimes (q')^* \mathcal{D}_Z \otimes \mathrm{Hom}((q')^* E, q^* E)). \tag{72}$$

Define $\tau K \in C^\infty_c(M; \Lambda^n \tau^* \otimes \mathcal{D})$ by

$$(\tau K)(m) \;=\; \int_{Z_m} \Phi(p) \; \mathrm{tr}_s \, K(p|p). \tag{73}$$

**Proposition 2** *Let $\rho$ be a linear functional on $C^\infty_c(M; \Lambda^n \tau^* \otimes \mathcal{D})$ Suppose that the linear functional $\eta$ on $\Omega^n \mathcal{B}$, defined by*

$$\eta(\phi) \;=\; \rho(\phi\big|_M), \tag{74}$$

*is a graded trace on $\Omega^* \mathcal{B}$. Then $\rho \circ \tau$ is a graded trace on $\mathrm{Hom}^\infty_{\mathcal{B}}(\mathcal{E}, \Omega^* \mathcal{E})$.*

**PROOF.** The proof is similar to that of Proposition 1. We omit the details.

**Proposition 3** *Let $\rho$ be a linear functional on $C^\infty_c(M; \Lambda^n \tau^* \otimes \mathcal{D})$ Suppose that the linear functional $\eta$ on $\Omega^n \mathcal{B}$, defined by*

$$\eta(\phi) \;=\; \rho(\phi\big|_M), \tag{75}$$

*is a closed graded trace on $\Omega^* \mathcal{B}$. Then $\rho \circ \tau$ annihilates $[\nabla, K]$ for all $K \in \mathrm{Hom}^\infty_{\mathcal{B}}(\mathcal{E}, \Omega^{n-1} \mathcal{E})$.*

**PROOF.** It suffices to show that

$$(\rho \circ \tau)([\nabla^{\mathcal{E}}, K]) \;=\; \eta\left(d^H(\tau(K))\right). \tag{76}$$

Let $\nabla^{\mathcal{E}_0} : C^\infty_c(P) \to C^\infty_c(P; \pi^* \tau^*)$ be differentiation in the $T^H P$-direction. It follows from (73) that

$$\begin{aligned}(d^H(\tau K))(m) \;=\; & \int_{Z_m} \Phi(p) \; \mathrm{tr}_s [\nabla^{\mathcal{E}}, K](p|p) \,+ \\ & \int_{Z_m} \nabla^{\mathcal{E}_0} \Phi(p) \;\wedge\; \mathrm{tr}_s \, K(p|p).\end{aligned} \tag{77}$$

Now $\eta\left(\int_{Z_m} \nabla^{\mathcal{E}_0} \Phi(p) \;\wedge\; \mathrm{tr}_s \, K(p|p)\right)$ can be written as $\int_P \nabla^{\mathcal{E}_0} \Phi \wedge \mathcal{O}$ for some $G$-invariant $\mathcal{O}$. From (57), $\int_{G^{\pi(p)}} \nabla^{\mathcal{E}_0} \Phi(pg) = 0$. Then decomposing the measure on $P$ with respect to $P \to P/G$ gives that $\int_P \nabla^{\mathcal{E}_0} \Phi \wedge \mathcal{O} \;=\; 0$. Equation (76) follows.

**Example 7 :** Following the notation of Example 6, let $c$ be a closed holonomy-invariant transverse $n$-current for $\mathcal{F}$. Let $c_i$ be the $n$-current on $V_i$ which is the restriction of $c$. Let $\{\phi_i\}_{i=1}^N$ be a partition of unity that is subordinate to $\{U_i\}_{i=1}^N$. For $\omega \in C^\infty_c(M; \Lambda^n \tau^* \otimes \mathcal{D})$, put $\rho(\omega) \;=\; \sum_{i=1}^N \langle \left(\int_{W_i} \phi_i \, \omega\right), \; c_i \rangle$. Then $\rho$ satisfies the hypotheses of Proposition 3.

### 4.3 The $s \to 0$ limit of the superconnection Chern character

In this subsection we extend $\mathrm{End}^\infty(\mathcal{E})$ to an rapid-decay algebra $\mathrm{End}^\omega(\mathcal{E})$. Given a $G$-invariant Dirac-type operator acting on sections of $E$, we consider the Bismut superconnections $\{A_s\}_{s>0}$ on $\mathcal{E}$. We compute the $s \to 0$ limit of the pairing between the Chern character of $A_s$ and a closed graded trace on $\Omega^*\mathcal{B}$ that is concentrated on the units $M$.

We now choose a $G$-invariant vertical Riemannian metric $g^{TZ}$ on the submersion $\pi : P \to M$ and a $G$-invariant horizontal distribution $T^H P$. Given $m \in M$, let $d_m$ denote the corresponding metric on $Z_m$. We note that $\{Z_m\}_{m\in M}$ has uniformly bounded geometry.

Let $\mathrm{End}^\omega_{\mathcal{B}}(\mathcal{E})$ be the algebra formed by $G$-invariant operators $K$ as in (56) whose integral kernels $K(p'|p) \in C^\infty(P \times_M P; (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E))$ are such that for all $q \in \mathbb{Z}^+$,

$$\sup_{(p',p)\in P\times_M P} e^{q\, d(p',p)}\, |K(p'|p)| \, < \, \infty, \tag{78}$$

along with the analogous property for the covariant derivatives of $K$.

**Proposition 4** *Let $\rho$ be a linear functional on $C^\infty_c(M;\mathcal{D})$. Suppose that the linear functional $\eta$ on $\mathcal{B}$, defined by*

$$\eta(b) \, = \, \rho(b\big|_M), \tag{79}$$

*is a trace on $\mathcal{B}$. Then $\rho \circ \tau$ is a supertrace on $\mathrm{End}^\omega_{\mathcal{B}}(\mathcal{E})$.*

**PROOF.** The proof is formally the same as that of Proposition 1. We omit the details

Let $\mathrm{Hom}^\omega_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$ be the algebra formed by $G$-invariant operators $K$ as in (71) whose integral kernels

$$K(p'|p) \in C^\infty_c(P \times_M P; \Lambda^*((\pi \circ q)^*\tau^*) \otimes (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E)) \tag{80}$$

are such that for all $q \in \mathbb{Z}^+$,

$$\sup_{(p',p)\in P\times_M P} e^{q\, d(p',p)}\, |K(p'|p)| \, < \, \infty, \tag{81}$$

along with the analogous property for the covariant derivatives of $K$.

**Proposition 5** *Let $\rho$ be a linear functional on $C^\infty_c(M;\Lambda^n\tau^* \otimes \mathcal{D})$ Suppose that the linear functional $\eta$ on $\Omega^n\mathcal{B}$, defined by*

$$\eta(\phi) \, = \, \rho(\phi\big|_M), \tag{82}$$

*is a graded trace on $\Omega^*\mathcal{B}$. Then $\rho \circ \tau$ is a graded trace on $\mathrm{Hom}^\omega_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$.*

**PROOF.** The proof is formally the same as that of Proposition 2. We omit the details.

**Proposition 6** *Let $\rho$ be a linear functional on $C_c^\infty(M;\Lambda^n\tau^*\otimes\mathcal{D})$ Suppose that the linear functional $\eta$ on $\Omega^n\mathcal{B}$, defined by*

$$\eta(\phi) \;=\; \rho(\phi\big|_M), \tag{83}$$

*is a closed graded trace on $\Omega^*\mathcal{B}$. Then $\rho\circ\tau$ annihilates $[\nabla, K]$ for all $K\in\mathrm{Hom}^\omega_{\mathcal{B}}(\mathcal{E},\Omega^{n-1}\mathcal{E})$.*

**PROOF.** The proof is formally the same as that of Proposition 3. We omit the details.

Suppose that $Z$ is even-dimensional. Let $E$ be a $G$-invariant Clifford bundle on $P$ which is equipped with a $G$-invariant connection. For simplicity of notation, we assume that $E = S^Z\,\widehat{\otimes}\,\widetilde{V}$, where $S^Z$ is a vertical spinor bundle and $\widetilde{V}$ is an auxiliary vector bundle on $P$. More precisely, suppose that the vertical tangent bundle $TZ$ has a $G$-invariant spin structure. Let $S^Z$ be the vertical spinor bundle, a $G$-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on $P$. Let $\widetilde{V}$ be another $G$-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on $P$ which is equipped with a $G$-invariant Hermitian connection. That is, $\widetilde{V}$ is the pullback of a Hermitian vector bundle $G$ on $P/G$ with a Hermitian connection $\nabla^V$. Then we put $E = S^Z\,\widehat{\otimes}\,\widetilde{V}$. The case of general $G$-invariant Clifford bundles $E$ can be treated in a way completely analogous to what follows.

Let $\nabla^{TZ}$ be the Bismut connection on $TZ$, as constructed using the horizontal distribution $(d\mu)^{-1}(T^H(P/G))$ on $P$; see, for example, Berline-Getzler-Vergne [5, Proposition 10.2]. The $G$-invariance of $\nabla^{TZ}$ and $\nabla^{\widetilde{V}}$ implies that $\widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})$ lies in $C^\infty(P;\Lambda^*(TZ)^*\otimes\Lambda^*(\pi^*\tau^*))$.

Let $Q\in\mathrm{End}_{\mathcal{B}}(\mathcal{E})$ denote the vertical Dirac-type operator. From finite-propagation-speed estimates as in Lott [22, Proof. of Prop 8], along with the bounded geometry of $\{Z_m\}_{m\in M}$, for any $s > 0$ we have

$$e^{-\,s^2\,Q^2}\in\mathrm{End}^\omega_{\mathcal{B}}(\mathcal{E}). \tag{84}$$

Let $A_s : \mathcal{E}\to\Omega^*\mathcal{E}$ be the superconnection

$$A_s \;=\; s\,Q \,+\, \nabla^{\mathcal{E}} \,-\, \frac{1}{4s}\,c(T^P). \tag{85}$$

Here $c(T^P)$ is Clifford multiplication by the curvature 2-form $T^P$ of $(d\mu)^{-1}(T^H(P/G))$, restricted to the horizontal vectors $T^HP$. We note that the analogous connection term of the Bismut superconnection [5, Proposition 10.15] has an additional term to make it Hermitian, but in our setting this term is incorporated into the horizontal differentiation of the vertical density. One can use finite-propagation-speed estimates, along with the bounded geometry of $\{Z_m\}_{m\in M}$ and the Duhamel expansion as in [5, Theorem 9.48], to show that we obtain a well-defined element $e^{-\,(A^s)^2\,-\,\mathcal{L}_{\widetilde{T}}}$ of $\mathrm{Hom}^\omega_{\mathcal{B}}(\mathcal{E},\Omega^*\mathcal{E})$; see [18, Theorem 3.1] for an analogous statement when $P = G = G_{hol}$.

Let $\mathcal{R}$ be the rescaling operator which, for $p$ even, multiplies a $p$-form by $(2\pi i)^{-\frac{p}{2}}$. Put

$$\mathrm{ch}(A_s) \;=\; \mathcal{R}\left(\tau e^{-\,A_s^2\,-\,\mathcal{L}_{\widetilde{T}}}\right) \in C_c^\infty(M;\Lambda^*\tau^* \,\otimes\, \mathcal{D}). \tag{86}$$

**Theorem 2** *Given a linear functional $\rho$ which satisfies the hypotheses of Proposition 6,*

$$\lim_{s\to 0}\rho(\mathrm{ch}(A_s)) \;=\; \rho\left(\int_Z \Phi\,\widehat{A}(\nabla^{TZ})\;\mathrm{ch}(\nabla^{\widetilde{V}})\right). \tag{87}$$

**PROOF.** Using Lemmas 13 and 14, $A_s^2 + \mathcal{L}_{\widetilde{T}}$ is $G$-invariant. Let $A_s'$ be the corresponding Bismut superconnection on the foliated manifold $P/G$, a locally-defined differential operator constructed using the horizontal distribution $T^H(P/G)$. By construction, $A_s^2 + \mathcal{L}_{\widetilde{T}}$ is the pullback under $\mu$ of $(A_s')^2$, where we use the identification $\Lambda^*(\pi^*\tau^*) = \mu^*\Lambda^*(NF)^*$. From [5, Theorem 10.23], the $s \to 0$ limit of the supertrace of the kernel of $e^{-\,(A_s')^2}$, when restricted to the diagonal of $(P/G)\times(P/G)$, is $\widehat{A}(\nabla^{TF})\;\mathrm{ch}(\nabla^V)$. Then the $s \to 0$ limit of the supertrace of the kernel of $e^{-\,A_s^2\,-\,\mathcal{L}_{\widetilde{T}}}$, when restricted to the diagonal of $P \times P$, is the pullback under $\mu$ of $\widehat{A}(\nabla^{TF})\;\mathrm{ch}(\nabla^V)$, i.e. $\widehat{A}(\nabla^{TZ})\;\mathrm{ch}(\nabla^{\widetilde{V}})$. The theorem follows.

**Remark :** If $P = G = G_{hol}$ then an analogue of Theorem 2 appears in [18, Theorem 2.1].

If we put

$$G' \;=\; \{(p_1,p_2)\in P\times P \;:\; \pi(p_1) \;=\; \pi(p_2)\}/G. \tag{88}$$

then $G'$ has the structure of a foliation groupoid, with units $G'^{(0)} = P/G$. In this way we could reduce from the case of $G$ acting on $P$ to the case of the foliation groupoid $G'$ acting on itself. However, doing so would not really simplify any of the constructions.

## 5 Index Theorem

In this section we prove the main result of the paper, Theorem 5.

### *5.1 The index class*

In this subsection we construct the index class $\mathrm{Ind}(D)\in \mathrm{K}_0(\mathfrak{A})$. We describe its pairing with a closed graded trace on $\mathcal{B}$. We prove that the pairing of $\mathrm{Ind}(D)$ with the closed graded trace equals the pairing of $\mathrm{ch}(A_s)$ with the closed graded trace.

Consider the algebra $\mathfrak{A} = \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$. Let $D : \mathcal{E}^+ \to \mathcal{E}^-$ be the restriction of $Q$ to the positive subspace $\mathcal{E}^+$ of $\mathcal{E}$. We construct an index projection following Connes-Moscovici [12] and Moscovici-Wu [23]. Let $u \in C^\infty(\mathbb{R})$ be an even function such that $w(x) = 1 - x^2\,u(x)$

is a Schwartz function and the Fourier transforms of $u$ and $w$ have compact support [23, Lemma 2.1]. Define $\overline{u} \in C^\infty([0,\infty))$ by $\overline{u}(x) = u(x^2)$. Put $\mathcal{P} = \overline{u}(D^*D)D^*$, which we will think of as a parametrix for $D$, and put $S_+ = I - \mathcal{P}D$, $S_- = I - D\mathcal{P}$. Consider the operator

$$L = \begin{pmatrix} S_+ & -(I+S_+)\mathcal{P} \\ D & S_- \end{pmatrix}, \tag{89}$$

with inverse

$$L^{-1} = \begin{pmatrix} S_+ & \mathcal{P}(I+S_-) \\ -D & S_- \end{pmatrix}. \tag{90}$$

The index projection is defined by

$$p = L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} S_+^2 & S_+(I+S_+)\mathcal{P} \\ S_-D & I - S_-^2 \end{pmatrix}. \tag{91}$$

Put

$$p_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{92}$$

By definition, the index of $D$ is

$$\mathrm{Ind}(D) = [p - p_0] \in \mathrm{K}_0(\mathfrak{A}). \tag{93}$$

As $Q$ is $G$-invariant, the operator $l$ of (46) commutes with $p$, and (47) holds for $\xi \in \mathrm{Im}(p)$. If $\rho$ is a linear functional which satisfies the hypotheses of Proposition 3, define the pairing of $\rho$ with $\mathrm{Ind}(D)$ by

$$\begin{aligned}\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho \rangle &= (2\pi i)^{-\,deg(\rho)/2} \\ &\quad \rho\left(\tau\left(p\, e^{-(p\circ\nabla^{\mathcal{E}}\circ p)^2 - \mathcal{L}_{\widetilde{T}}}\, p - p_0\, e^{-(p_0\circ\nabla^{\mathcal{E}}\circ p_0)^2 - \mathcal{L}_{\widetilde{T}}}\, p_0\right)\right),\end{aligned} \tag{94}$$

where we have extended the ungraded trace $\tau$ in the obvious way to act on $(2\times 2)$-matrices. (See [16, Section 5] for the justification of the definition.)

**Theorem 3** *For all $s > 0$,*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho \rangle = \rho(\mathrm{ch}(A_s)). \tag{95}$$

**PROOF.** The proof follows the lines of the proof of [16, Proposition 4 and Theorem 3], to which we refer for details. We only present the main idea. Put

$$\nabla' = \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} \circ \nabla^{\mathcal{E}} \circ L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \nabla^{\mathcal{E}} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right). \tag{96}$$

Then one can show algebraically that

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho \rangle \ = \ \rho \left( \mathcal{R} \, \tau e^{-(\nabla')^2 - \mathcal{L}_{\widetilde{T}}} \right), \tag{97}$$

where the $\tau$ on the right-hand-side is now a graded trace. Next, one shows that

$$\rho \left( \mathcal{R} \, \tau e^{-(\nabla')^2 - \mathcal{L}_{\widetilde{T}}} \right) \ = \ \rho(\mathrm{ch}(A_s)) \tag{98}$$

by performing a homotopy from $\nabla'$ to $A_s$, from which the theorem follows.

### *5.2 Construction of $\omega_\rho$*

In this subsection we construct the universal class $\omega_\rho \in \mathrm{H}^*(BG; o)$. We express $\rho(\mathrm{ch}(A_s))$ as an integral involving the pullback of $\omega_\rho$.

Put $V \ = \ \widetilde{V}/G$, a Hermitian vector bundle on $P/G$ with a compatible connection $\nabla^V$.

Let $o(\tau)$ be the orientation bundle of $\tau$, a flat real line bundle on $M$. Let $\rho$ satisfy the hypotheses of Proposition 3. By duality, $\rho$ corresponds to a closed distributional form $*\rho \in \Omega^{\dim(M)-n}(M; o(\tau))$.

Let $EG$ denote the bar construction of a universal space on which $G$ acts freely. That is, put

$$G^{(n)} \ = \ \{(g_1, \ldots, g_n) \ : \ s(g_1) \ = \ r(g_2), \ldots, s(g_{n-1}) \ = \ r(g_n)\}. \tag{99}$$

Then $EG$ is the geometric realization of a simplicial manifold given by $E_nG \ = \ G^{(n+1)}$, with face maps

$$d_i(g_0, \ldots, g_n) \ = \ \begin{cases} (g_1, \ldots, g_n) & \text{if } i = 0, \\ (g_0, \ldots, g_{i-1}g_i, \ldots, g_n) & \text{if } 1 \ \leq \ i \ \leq \ n \end{cases} \tag{100}$$

and degeneracy maps

$$s_i(g_0, \ldots, g_n) \ = \ (g_0, \ldots, g_i, 1, g_{i+1}, \ldots g_n), \quad 0 \ \leq \ i \ \leq \ n. \tag{101}$$

Here 1 denotes a unit. The action of $G$ on $EG$ is induced from the action on $E_nG$ given by $(g_0, \ldots, g_n) \, g \ = \ (g_0, \ldots, g_n g)$. Let $BG$ be the quotient space. Define $\pi' \ : \ EG \rightarrow M$ as the extension of $(g_0, \ldots, g_n) \rightarrow s(g_n)$. Put $\Omega^{n_1, n_2}(EG) \ = \ \Omega^{n_1}(G^{(n_2+1)})$ and $\Omega^{n_1, n_2}(BG) \ = \ (\Omega^{n_1, n_2}(EG))^G$. Let $\Omega^*(BG)$ be the total complex of $\Omega^{*,*}(BG)$. Here the forms on $G^{(n_2+1)}$ can be either smooth or distributional, depending on the context. We will speak correspondingly of smooth or distributional elements of $\Omega^*(BG)$. In either case, the cohomology of $\Omega^*(BG)$ equals $\mathrm{H}^*(BG; \mathbb{R})$. There is a similar discussion for twistings by a local system.

The action of $G$ on $P$ is classified by a continuous $G$-equivariant map $\widehat{\nu} \ : \ P \rightarrow EG$. Let

$\nu : P/G \to BG$ be the $G$-quotient of $\widehat{\nu}$. There are commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\widehat{\nu}} & EG \\ \pi \big\downarrow & & \pi' \big\downarrow \\ M & \xrightarrow{\mathrm{Id}.} & M \end{array} \tag{102}$$

and

$$\begin{array}{ccc} P & \xrightarrow{\widehat{\nu}} & EG \\ \big\downarrow & & \big\downarrow \\ P/G & \xrightarrow{\nu} & BG. \end{array} \tag{103}$$

As $P/G$ is compact, we may assume that $\nu$ is Lipschitz.

Consider $(\pi')^*(*\rho) \in \Omega^*(EG; (\pi')^* o(\tau))$, a closed distributional form on $EG$. Let $o$ be the $G$-quotient of $(\pi')^* o(\tau)$, a flat real line bundle on $BG$. Then $(\pi')^*(*\rho)$ pulls back from a closed distributional form in $\Omega^*(BG; o)$, which represents a class in $\mathrm{H}^*(BG; o)$. Let $\omega_\rho \in \Omega^*(BG; o)$ be a closed smooth form representing the same cohomology class. Let $\widehat{\omega}_\rho \in \Omega^*(EG; (\pi')^* o(\tau))$ be its pullback to $EG$. As $\nu$ is Lipschitz, $\nu^* \omega_\rho$ is an $L^\infty$-form on $P/G$.

**Theorem 4**

$$\rho\left(\int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right) = \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\, \nu^* \omega_\rho. \tag{104}$$

**PROOF.** Let $*\left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)$ be the dual of $\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})$. We will think of $*\left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)$ as a cycle on $P$ and $(\pi')^*(*\rho)$ as a cocycle on $EG$. Then

$$\begin{aligned} \rho\left(\int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right) &= \langle \pi_* \left( * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), *\rho\rangle_M \\ &= \langle * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right), \pi^*(*\rho)\rangle_P \\ &= \langle \widehat{\nu}_* \left( * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), (\pi')^*(*\rho)\rangle_{EG} \\ &= \langle \widehat{\nu}_* \left( * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), \widehat{\omega}_\rho\rangle_{EG} \\ &= \langle * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right), \widehat{\nu}^* \widehat{\omega}_\rho\rangle_P \\ &= \int_P \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\, \widehat{\nu}^* \widehat{\omega}_\rho \\ &= \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\, \nu^* \omega_\rho. \end{aligned} \tag{105}$$

**Remark :** If one were willing to work with orbifolds $P/G$ instead of manifolds then one could extend Theorem 4 to general proper cocompact actions, with $\omega_\rho \in \mathrm{H}^*(\underline{B}G; o)$ being a cohomology class on the classifying space for proper $G$-actions.

### *5.3 Proof of index theorem*

**Theorem 5** *If $G$ acts freely, properly discontinuously and cocompactly on $P$ and $\rho$ satisfies the hypotheses of Proposition 6 then*

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle \; = \; \int_{P/G} \widehat{A}(TF) \; \mathrm{ch}(V) \, \nu^* \omega_\rho. \tag{106}$$

**PROOF.** If $Z$ is even-dimensional then the claim follows from Theorems 2, 3 and 4. If $Z$ is odd-dimensional then one can reduce to the even-dimensional case by a standard trick involving taking the product with a circle.

**Example 8 :** Suppose that $(M, \mathcal{F})$ is a closed foliated manifold. Take $P = G = G_{hol}$. Let $\mu$ be a holonomy-invariant transverse measure for $\mathcal{F}$. Take $\rho$ as in Example 6. Then Theorem 5 reduces to Connes' $L^2$-foliation index theorem [11, Section I.5.$\gamma$, Theorem 7]

$$\langle \mathrm{Ind}\, D, \rho \rangle \; = \; \langle \widehat{A}(TF) \; \mathrm{ch}(V), RS_\mu \rangle, \tag{107}$$

where $RS_\mu$ is the Ruelle-Sullivan current associated to $\mu$ [11, Section I.5.$\beta$].

**Example 9 :** Let $(M, \mathcal{F})$ be a closed manifold equipped with a codimension-$q$ foliation. Take $P = G = G_{hol}$. Let $\mathrm{H}^*(\mathrm{Tr}\, \mathcal{F})$ denote the Haefliger cohomology of $(M, \mathcal{F})$ [17]. Recall that there is a linear map $\int_{\mathcal{F}} : \mathrm{H}^*(M) \to \mathrm{H}^{*-n+q}(\mathrm{Tr}\, \mathcal{F})$. Let $c$ be a closed holonomy-invariant transverse current for $\mathcal{F}$. Take $\rho$ as in Example 7. Then Theorem 5 becomes

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle \; = \; \langle \int_{\mathcal{F}} \widehat{A}(TF) \; \mathrm{ch}(V), c \rangle. \tag{108}$$

This is a consequence of the Connes-Skandalis foliation index theorem, along with the result of Connes that $\rho$ gives a higher trace on the reduced foliation $C^*$-algebra; see [4,10,13].

**Example 10 :** Let $M$ be a closed oriented $n$-dimensional manifold. Let $G = M$ be the groupoid that just consists of units. Let $P$ be a closed manifold that is the total space of an oriented fiber bundle $\pi : P \to M$ with fiber $Z$. Let $c$ be a closed current on $M$ with homology class $[c] \in \mathrm{H}_*(M; \mathbb{C})$. With $* : \mathrm{H}_*(M; \mathbb{C}) \to \mathrm{H}^{n-*}(M; \mathbb{C})$ being the Poincaré isomorphism, Theorem 5 becomes

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), c \rangle \; = \; \int_P \widehat{A}(TZ) \; \mathrm{ch}(V) \, \pi^*(*[c])). \tag{109}$$

This is a consequence of the Atiyah-Singer families index theorem [2], as the right-hand-side equals $\langle \int_Z \widehat{A}(TZ)\ \mathrm{ch}(V), c\rangle$.

**Example 11 :** Let $G$ be a discrete group that acts freely, properly discontinuously and cocompactly on a manifold $P$. As its space of units $M$ is a point, let $\rho$ be the identity map $C^\infty(M) \to \mathbb{C}$. Then Theorem 5 reduces to Atiyah's $L^2$-index theorem [1]

$$\langle \mathrm{Ind}\, D, \rho \rangle \ = \ \int_{P/G} \widehat{A}(TP/G)\ \mathrm{ch}(V). \tag{110}$$

# A Appendix

This is an addendum to [16], in which we use finite propagation speed methods to improve [16, Theorem 3]. In the improved version we allow $\eta$ to be a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$, as opposed to $\Omega^*(B, \mathcal{B}^\omega)$. There is a similar improvement of [16, Theorem 6].

We will follow the notation of [16].

## *A.1 Finite propagation speed*

Let $f \in C_c^\infty(\mathbb{R})$ be a smooth even function with support in $[-\epsilon, \epsilon]$. Put

$$\widehat{f}(y) \ = \ \int_{\mathbb{R}} f(x)\ \cos(xy)\, dx, \tag{A.1}$$

a smooth even function. With $A_s$ as in [16, (4.7)], put

$$\widehat{f}(A_s) \ = \ \int_{\mathbb{R}} f(x)\ \cos\left(x\, A_s\right)\ dx. \tag{A.2}$$

Let us describe $\cos\left(x\, A_s\right)$ explicitly, using the fact that it satisfies

$$\left(\partial_x^2 \ + \ A_s^2\right)\ \cos\left(x\, A_s\right) \ = \ 0. \tag{A.3}$$

Write $A_s^2 \ = \ s^2 Q^2 \ + \ X$. We first consider a solution $u(\cdot, x)$ of the inhomogeneous wave equation

$$\left(\partial_x^2 \ + \ s^2 Q^2\right)\ u \ = \ f \tag{A.4}$$

with initial conditions $u(\cdot, 0) = u_0(\cdot)$ and $u_x(\cdot, 0) = 0$. Then $u(\cdot, x)$ is given by

$$u(x) \ = \ \cos(xsQ) u_0 \ + \ \int_0^x \frac{\sin((x-v)sQ)}{sQ}\ f(v)\, dv. \tag{A.5}$$

Putting $f = -Xu$ and iterating, we obtain an expansion of $\cos(x A_s)$ of the form

$$\cos(x A_s) = \cos(xsQ) - \int_0^x \frac{\sin((x-v)sQ)}{sQ} X \cos(vsQ)\, dv + \ldots \tag{A.6}$$

Because $X$ has positive form degree, there is no problem with the convergence of the series.

From finite propagation speed results, we know that $\cos(xsQ)$ has a Schwartz kernel $\cos(xsQ)(p'|p)$ with support on $\{(p',p) : d(p',p) \le xs\}$, and similarly for $\frac{\sin(xsQ)}{sQ}$; see Taylor [29, Chapter 4.4]. Using the compactness of $h$, it follows that the $(m,n)$-component $\widehat{f}(A_s)_{(m,n)}$ lies in $\mathrm{Hom}^{\infty}_{C_c^\infty(B)\rtimes\Gamma}(C_c^\infty(\widehat{M};\widehat{E}), \Omega^{m,n}(B,\mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\widehat{M};\widehat{E}))$.

Finally, define $\mathrm{ch}_{\widehat{f}}(A_s) \in \Omega^*(B,\mathbb{C}\Gamma)_{ab}$ by

$$\mathrm{ch}_{\widehat{f}}(A_s) = \mathcal{R}\,\mathrm{Tr}_{s,\langle e\rangle}\, \widehat{f}(A_s). \tag{A.7}$$

### *A.2 Index Pairing*

In this subsection we show that for all $s > 0$ and all closed graded traces $\eta$ on $\Omega^*(B,\mathbb{C}\Gamma)$, $\langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle = \langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle$. The method of proof is essentially the same as that of [16, Section 5], which in turn was inspired by Nistor [25].

In analogy to [16, Section 5.3], put $\mathcal{E} = C_c^\infty(\widehat{M};\widehat{E})$ and $\widetilde{\mathfrak{A}} = \mathrm{End}^{\infty}_{C_c^\infty(B)\rtimes\Gamma}\left(C_c^\infty(\widehat{M};\widehat{E})\right)$. Let $D : \mathcal{E}^+ \to \mathcal{E}^-$ be the restriction of $Q$ to the positive subspace $\mathcal{E}^+$ of $\mathcal{E}$. We construct an index projection following [12] and [23]. Let $u \in C^\infty(\mathbb{R})$ be an even function such that $w(x) = 1 - x^2 u(x)$ is a Schwartz function and the Fourier transforms of $u$ and $w$ have compact support [23, Lemma 2.1]. Define $\overline{u} \in C^\infty([0,\infty))$ by $\overline{u}(x) = u(x^2)$. Put $\mathcal{P} = \overline{u}(D^*D)D^*$, which we will think of as a parametrix for $D$, and put $S_+ = I - \mathcal{P}D$, $S_- = I - D\mathcal{P}$. Consider the operator

$$L = \begin{pmatrix} S_+ & -(I+S_+)\mathcal{P} \\ D & S_- \end{pmatrix}, \tag{A.8}$$

with inverse

$$L^{-1} = \begin{pmatrix} S_+ & \mathcal{P}(I+S_-) \\ -D & S_- \end{pmatrix}. \tag{A.9}$$

The index projection is defined by

$$p = L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} S_+^2 & S_+(I+S_+)\mathcal{P} \\ S_-D & I - S_-^2 \end{pmatrix}. \tag{A.10}$$

Put

$$p_0 \;=\; \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{A.11}$$

By definition, the index of $D$ is

$$\mathrm{Ind}(D) \;=\; [p \;-\; p_0] \in \mathrm{K}_0(\widetilde{\mathfrak{A}}). \tag{A.12}$$

Put $\widetilde{\Omega}^* \;=\; \mathrm{Hom}^\infty_{C_c^\infty(B)\rtimes\Gamma}(C_c^\infty(\widehat{M};\widehat{E}), \Omega^*(B,\mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\widehat{M};\widehat{E}))$, a graded algebra with derivation $\nabla \;=\; \nabla^{(1,0)} + \nabla^{(0,1)}$. If $\eta$ is a closed graded trace on $\Omega^*(B,\mathbb{C}\Gamma)$, define the pairing of $\eta$ with $\mathrm{Ind}(D)$ by

$$\langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle \;=\; (2\pi i)^{-\,deg(\eta)/2}\, \langle \mathrm{Tr}_{\langle e\rangle} \left( \widehat{f}(p\circ\nabla\circ p) \;-\; \widehat{f}(p_0\circ\nabla\circ p_0)\right), \eta\rangle. \tag{A.13}$$

(See [16, Section 5] for the justification of the definition.)

**Theorem 6** *For all $s > 0$,*

$$\langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle \;=\; \langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle. \tag{A.14}$$

**PROOF.** The proof follows the lines of the proof of [16, Proposition 4 and Theorem 3], to which we refer for details. We only present the main idea. Put

$$\nabla' \;=\; \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1}\circ\nabla\circ L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) \;+\; \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \nabla \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right). \tag{A.15}$$

Then one can show algebraically that

$$\langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle \;=\; \langle \mathcal{R}\ \mathrm{Tr}_{s,\langle e\rangle}\, \widehat{f}(\nabla'), \eta\rangle. \tag{A.16}$$

Next, one shows that

$$\langle \mathcal{R}\ \mathrm{Tr}_{s,\langle e\rangle}\, \widehat{f}(\nabla'), \eta\rangle \;=\; \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle \tag{A.17}$$

by performing a homotopy from $\nabla'$ to $A_s$, from which the theorem follows. The argument is the same as in the proof of [16, Proposition 4]. We refer to [16], and will only indicate the necessary modifications of the equations in [16, Section 5.2].

As in [16, (5.20)], for $t \in [0,1]$ put

$$A(t) \;=\; \begin{pmatrix} (\nabla')^+ & t\,D^* \\ t\,D & (\nabla')^- \end{pmatrix}. \tag{A.18}$$

The analog of [16, (5.26)] is

$$\cos\left(x\, A(t)\right) \equiv \tag{A.19}$$
$$\begin{pmatrix} \cos\left(x\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right) & \mathcal{Z} \\ 0 & D\,\cos\left(x\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right)\,\mathcal{P} \end{pmatrix},$$

where

$$\mathcal{Z} \;=\; -\int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right)}{\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}} \tag{A.20}$$
$$\left(t\,[(\nabla')^-, D^*] \,+\, t((\nabla')^+ \,-\, (\nabla')^-)D^*\right) \cos\left(v\,\sqrt{(\nabla^-)^2 \,+\, t^2\, DD^*}\right)\, dv$$

and the left-hand-side of (A.19) is to be multiplied by $f$ and then integrated. As in [16, (5.30)],

$$\frac{dA}{dt} \;=\; \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix} \tag{A.21}$$

The analog of [16, (5.31)] is

$$\mathrm{Tr}_s\left(\frac{dA}{dt}\begin{pmatrix} \cos\left(x\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right) & \mathcal{Z} \\ 0 & D\,\cos\left(x\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right)\,\mathcal{P} \end{pmatrix}\right) \tag{A.22}$$
$$=\; -\;\mathrm{Tr}\left(D\,\mathcal{Z}\right) \;=$$
$$t\;\mathrm{Tr}\left(D\int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}\right)}{\sqrt{((\nabla')^+)^2 \,+\, t^2\, D^*D}}\;\left([(\nabla')^-, D^*] \,+\, ((\nabla')^+ \,-\, (\nabla')^-)D^*\right)\right.$$
$$\left.\cos\left(v\,\sqrt{(\nabla^-)^2 \,+\, t^2\, DD^*}\right)\right)\, dv.$$

The analog of [16, (5.32)] is

$$D \int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 \ + \ t^2\, D^*D}\right)}{\sqrt{((\nabla')^+)^2 \ + \ t^2\, D^*D}} \ \left([(\nabla')^-, D^*] \ + \ ((\nabla')^+ \ - \ (\nabla')^-)D^*\right) \tag{A.23}$$
$$\cos\left(v\,\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)\ dv \ \equiv$$
$$\int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)}{\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}}\ D\ \left([(\nabla')^-, D^*] \ + \ ((\nabla')^+ \ - \ (\nabla')^-)D^*\right)$$
$$\cos\left(v\,\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)\ dv \ \equiv$$
$$\int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)}{\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}}\ [\nabla^-, DD^*]$$
$$\cos\left(v\,\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)\ dv.$$

The analog of [16, (5.33)] is

$$\mathrm{Tr}\left(\int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)}{\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}}\ [\nabla^-, DD^*] \right. \tag{A.24}$$
$$\left. \cos\left(v\,\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)\ dv\right) \ =$$
$$- \ t^{-2}\, d\ \mathrm{Tr}\left(\cos\left(x\,\sqrt{(\nabla^-)^2 \ + \ t^2\, DD^*}\right)\right).$$

The rest of the proof is as in [16, Proof of Proposition 4].

We define $\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta\rangle$ by formally taking $\widehat{f}(z) \ = \ e^{-z^2}$ in (A.13). This makes perfect sense, given that $\eta$ acts on elements of a fixed degree.

**Corollary 1** *a. The left-hand-side of (A.14) only depends on $f$ through the derivative $\widehat{f}^{(deg(\eta))}(0)$.*
*b. If $\widehat{f}^{(deg(\eta))}(0) \ = \ \left.\frac{d^{deg(\eta)} e^{-z^2}}{d^{deg(\eta)} z}\right|_{z=0}$ then*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta\rangle \ = \ \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle. \tag{A.25}$$

**PROOF.** a. From (A.13), the right-hand-side of (A.14) only depends on $f$ through the derivative $\widehat{f}^{(deg(\eta))}(0)$. From Theorem 6, the same must be true of the left-hand-side.
b. If $\widehat{f}^{(deg(\eta))}(0) \ = \ \left.\frac{d^{deg(\eta)} e^{-z^2}}{d^{deg(\eta)} z}\right|_{z=0}$ then $\widehat{f}$ has the same relevant term in its Taylor expansion as the function $z \to e^{-z^2}$, from which the corollary follows.

### A.3 Pairing of the Chern character of the index with general closed graded traces

In this subsection we prove a formula for the pairing of the Chern character of the index with a closed graded trace $\eta$ on $\Omega^*(B, \mathbb{C}\Gamma)$. The idea is to approximate the Gaussian function, which was previously used in forming the superconnection Chern character, by an appropriate function $\widehat{f}$.

**Theorem 7** *Given a closed graded trace $\eta$ on $\Omega^*(B, \mathbb{C}\Gamma)$,*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta \rangle \;=\; \langle \int_Z \Phi\, \widehat{A}(\nabla^{TZ})\; \mathrm{ch}(\nabla^{\widetilde{V}})\, e^{-\frac{\nabla^2_{can}}{2\pi i}}, \eta \rangle. \tag{A.26}$$

**PROOF.** Choose an even function $f \in C_c^\infty(\mathbb{R})$ so that $\widehat{f}$ satisfies the hypothesis of Corollary 1.b. By Corollary 1, it suffices to compute

$$\lim_{s\to 0} \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta \rangle. \tag{A.27}$$

With reference to (A.2), the local supertrace $\mathrm{tr}_s \cos(x\, A_s)\,(p,p)$ exists as a distribution in $x$. The singularities near $x = 0$ of the distribution have coefficients that are the same, up to constants, as the leading terms in the $x$-expansion of $\mathrm{tr}_s\, e^{-x^2 A_s^2}(p,p)$; see, for example, Sandoval [28] for the analogous statement for $\cos(xsQ)$. As in [5, Lemma 10.22], these are the terms that enter into the local index computation. Now $\cos(x\, A_s)$ satisfies (A.3), in analogy to the fact that $e^{-t\, A_s^2}$ satisfies the heat equation

$$\Big(\partial_t \,+\, A_s^2\Big)\; e^{-\,t\, A_s^2} \;=\; 0. \tag{A.28}$$

We can perform a Getzler rescaling as in the proof of [16, Theorem 2], to see that for the purposes of computing the local index, we can effectively replace the $A_s^2$-term in the differential operator of (A.3) by [16, (4.12)]. Thus we are reduced to considering the wave operator of the harmonic oscillator Hamiltonian. The rest of the proof of the theorem can in principle be carried out in a way similar to that of [16, Theorem 2]. However, we can shortcut the calculations by noting that Corollary 1, along with the choice of $f$, implies that the result of the local calculation must be the same as $\lim_{s\to 0}\langle \mathrm{ch}(A_s), \eta\rangle$, which was already calculated in [16, Theorem 2].

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