FORMALITY FOR ALGEBROID STACKS

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Abstract. We extend the formality theorem of M. Kontsevich from deformations of the structure sheaf on a manifold to deformations of gerbes.

1. Introduction

In the fundamental paper [11] M. Kontsevich showed that the set of equivalence classes of formal deformations the algebra of functions on a manifold is in one-to-one correspondence with the set of equivalence classes of formal Poisson structures on the manifold. This result was obtained as a corollary of the formality of the Hochschild complex of the algebra of functions on the manifold conjectured by M. Kontsevich (cf. [10]) and proven in [11]. Later proofs by a different method were given in [14] and in [5].

In this paper we extend the formality theorem of M. Kontsevich to deformations of gerbes on smooth manifolds, using the method of [5]. Let $X$ be a smooth manifold; we denote by $\mathcal{O}_X$ the sheaf of complex valued $C^\infty$ functions on $X$. For a twisted form $\mathcal{S}$ of $\mathcal{O}_X$ regarded as an algebroid stack (see Section 2.5) we denote by $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$ the de Rham class of $\mathcal{S}$. The main result of this paper establishes an equivalence of 2-groupoid valued functors of Artin $\mathbb{C}$-algebras between $\text{Def}(\mathcal{S})$ (the formal deformation theory of $\mathcal{S}$, see [2]) and the Deligne 2-groupoid of Maurer-Cartan elements of $L_\infty$-algebra of multivector fields on $X$ twisted by a closed three-form representing $[\mathcal{S}]_{dR}$:

Theorem 6.1. Suppose that $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$. Let $H$ be a closed 3-form on $X$ which represents $[\mathcal{S}]_{dR} \in H^3_{dR}(X)$. For any Artin algebra $R$ with maximal ideal $m_R$ there is an equivalence of 2-groupoids

$$\text{MC}^2(s(\mathcal{O}_X)_H \otimes m_R) \cong \text{Def}(\mathcal{S})(R)$$

natural in $R$.

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Here, $s(O_X)_H$ denotes the $L_\infty$-algebra of multivector fields with the trivial differential, the binary operation given by Schouten bracket, the ternary operation given by $H$ (see 5.3) and all other operations equal to zero. As a corollary of this result we obtain that the isomorphism classes of formal deformations of $\mathcal{S}$ are in a bijective correspondence with equivalence classes of the formal twisted Poisson structures defined by P. Severa and A. Weinstein in [13].

The proof of the Theorem proceeds along the following lines. As a starting point we use the construction of the Differential Graded Lie Algebra (DGLA) controlling the deformations of $\mathcal{S}$. This construction was obtained in [1, 2]. Next we construct a chain of $L_\infty$-quasi-isomorphisms between this DGLA and $s(O_X)_H$, using the techniques of [5]. Since $L_\infty$-quasi-isomorphisms induce equivalences of respective Deligne groupoids, the result follows.

The paper is organized as follows. Section 2 contains the preliminary material on jets and deformations. Section 3 describes the results on the deformations of algebroid stacks. Section 4 is a short exposition of [5]. Section 5 contains the main technical result of the paper: the construction of the chain of quasi-isomorphisms mentioned above. Finally, in Section 6 the main theorem is deduced from the results of Section 5.

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2. Preliminaries

2.1. Notations. Throughout this paper, unless specified otherwise, $X$ will denote a $C^\infty$ manifold. By $O_X$ we denote the sheaf of complex-valued $C^\infty$ functions on $X$. $A^*_X$ denotes the sheaf of differential forms on $X$, and $T_X$ the sheaf of vector fields on $X$. For a ring $K$ we denote by $K^\times$ the group of invertible elements of $K$.

2.2. Jets. Let $pr_i: X \times X \to X$, $i = 1, 2$, denote the projection on the $i^{th}$ factor. Let $\Delta_X: X \to X \times X$ denote the diagonal embedding. Let $I_X := \ker(\Delta_X^\ast)$.

For a locally-free $O_X$-module of finite rank $E$ let

$$\mathcal{J}^k_X(E) := (pr_1)_\ast \left( O_{X \times X}/T_X^{k+1} \otimes_{pr_2^{-1}O_X} pr_2^{-1}E \right);$$

$$\mathcal{J}^k_X := \mathcal{J}^k_X(O_X).$$

It is clear from the above definition that $\mathcal{J}^k_X$ is, in a natural way, a commutative algebra and $\mathcal{J}^k_X(E)$ is a $\mathcal{J}^k_X$-module.

Let
1^{(k)} : \mathcal{O}_X \to \mathcal{J}_X^k

denote the composition

\mathcal{O}_X \xrightarrow{\text{pr}_1^*} (\text{pr}_1)_* \mathcal{O}_{X \times X} \to \mathcal{J}_X^k

In what follows, unless stated explicitly otherwise, we regard \( \mathcal{J}_X^k(\mathcal{E}) \) as a \( \mathcal{O}_X \)-module via the map \( 1^{(k)} \).

Let \( j^k : \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E}) \)
denote the composition

\( \mathcal{E} \xrightarrow{\text{pr}_1^*} (\text{pr}_1)_* \mathcal{O}_{X \times X} \otimes \mathcal{E} \to \mathcal{J}_X^k(\mathcal{E}) \)

The map \( j^k \) is not \( \mathcal{O}_X \)-linear unless \( k = 0 \).

For \( 0 \leq k \leq l \) the inclusion \( \mathcal{I}_X^{l+1} \to \mathcal{I}_X^{k+1} \) induces the surjective map \( \pi_{l,k} : \mathcal{J}_X^l(\mathcal{E}) \to \mathcal{J}_X^k(\mathcal{E}) \). The sheaves \( \mathcal{J}_X^k(\mathcal{E}) \), \( k = 0, 1, \ldots \) together with the maps \( \pi_{l,k}, k \leq l \) form an inverse system. Let \( \mathcal{J}_X(\mathcal{E}) = \mathcal{J}_X^\infty(\mathcal{E}) := \varprojlim \mathcal{J}_X^k(\mathcal{E}) \). Thus, \( \mathcal{J}_X(\mathcal{E}) \) carries a natural topology.

The maps \( 1^{(k)} \) (respectively, \( j^k \)), \( k = 0, 1, 2, \ldots \) are compatible with the projections \( \pi_{l,k} \), i.e. \( \pi_{l,k} \circ 1^{(l)} = 1^{(k)} \) (respectively, \( \pi_{l,k} \circ j^l = j^k \)). Let \( 1 := \varprojlim 1^{(k)} \), \( j^\infty := \varprojlim j^k \).

Let

\[ d_1 : \mathcal{O}_{X \times X} \otimes_{\text{pr}_1^{-1}\mathcal{O}_X} \text{pr}_2^{-1}\mathcal{E} \to \text{pr}_1^{-1}\mathcal{A}_X^1 \otimes_{\text{pr}_1^{-1}\mathcal{O}_X} \mathcal{O}_{X \times X} \otimes_{\text{pr}_2^{-1}\mathcal{O}_X} \text{pr}_2^{-1}\mathcal{E} \]

denote the exterior derivative along the first factor. It satisfies

\[ d_1(\mathcal{I}_X^{k+1} \otimes_{\text{pr}_1^{-1}\mathcal{O}_X} \text{pr}_2^{-1}\mathcal{E}) \subset \text{pr}_1^{-1}\mathcal{A}_X^1 \otimes_{\text{pr}_1^{-1}\mathcal{O}_X} \mathcal{I}_X^k \otimes_{\text{pr}_2^{-1}\mathcal{O}_X} \text{pr}_2^{-1}\mathcal{E} \]

for each \( k \) and, therefore, induces the map

\[ d_1^{(k)} : \mathcal{J}_X^k(\mathcal{E}) \to \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}_X^{k-1}(\mathcal{E}) \]

The maps \( d_1^{(k)} \) for different values of \( k \) are compatible with the maps \( \pi_{l,k} \) giving rise to the canonical flat connection

\[ \nabla^{\text{can}} : \mathcal{J}_X(\mathcal{E}) \to \mathcal{A}_X^1 \otimes_{\mathcal{O}_X} \mathcal{J}_X(\mathcal{E}) \].
2.3. Deligne groupoids. In [4] P. Deligne and, independently, E. Getzler in [8] associated to a nilpotent DGLA \( g \) concentrated in degrees grater than or equal to \(-1\) the 2-groupoid, referred to as the Deligne 2-groupoid and denoted \( MC^2(g) \) in [1], [2] and below. The objects of \( MC^2(g) \) are the Maurer-Cartan elements of \( g \). We refer the reader to [8] (as well as to [2]) for a detailed description. The above notion was extended and generalized by E. Getzler in [7] as follows.

To a nilpotent \( L_\infty \)-algebra \( g \) Getzler associates a (Kan) simplicial set \( \gamma \_\bullet(g) \) which is functorial for \( L_\infty \) morphisms. If \( g \) is concentrated in degrees greater than or equal to \( 1 - l \), then the simplicial set \( \gamma \_\bullet(g) \) is an \( l \)-dimensional hypergroupoid in the sense of J.W. Duskin (see [6]) by [7], Theorem 5.4.

Suppose that \( g \) is a nilpotent \( L_\infty \)-algebra concentrated in degrees grater than or equal to \(-1\). Then, according to [6], Theorem 8.6 the simplicial set \( \gamma \_\bullet(g) \) is the nerve of a bigroupoid, or, a 2-groupoid in our terminology. If \( g \) is a DGLA concentrated in degrees grater than or equal to \(-1\) this 2-groupoid coincides with \( MC^2(g) \) of Deligne and Getzler alluded to earlier. We extend our notation to the more general setting of nilpotent \( L_\infty \)-algebras as above and denote by \( MC^2(g) \) the 2-groupoid furnished by [6], Theorem 8.6.

For an \( L_\infty \)-algebra \( g \) and a nilpotent commutative algebra \( m \) the \( L_\infty \)-algebra \( g \otimes m \) is nilpotent, hence the simplicial set \( \gamma \_\bullet(g \otimes m) \) is defined and enjoys the following homotopy invariance property ([7], Proposition 4.9, Corollary 5.11):

**Theorem 2.1.** Suppose that \( f: g \rightarrow h \) is a quasi-isomorphism of \( L_\infty \) algebras and let \( m \) be a nilpotent commutative algebra. Then the induced map

\[
\gamma \_\bullet(f \otimes \text{Id}): \gamma \_\bullet(g \otimes m) \rightarrow \gamma \_\bullet(h \otimes m)
\]

is a homotopy equivalence.

2.4. Algebroid stacks. Here we give a very brief overview, referring the reader to [3, 9] for the details. Let \( k \) be a field of characteristic zero, and let \( R \) be a commutative \( k \)-algebra.

**Definition 2.2.** A stack in \( R \)-linear categories \( C \) on \( X \) is an \( R \)-algebroid stack if it is locally nonempty and locally connected, i.e. satisfies

1. any point \( x \in X \) has a neighborhood \( U \) such that \( C(U) \) is nonempty;
2. for any \( U \subseteq X, x \in U, A, B \in C(U) \) there exits a neighborhood \( V \subseteq U \) of \( x \) and an isomorphism \( A|_V \cong B|_V \).

For a prestack \( C \) we denote by \( \tilde{C} \) the associated stack.
For a category $C$ denote by $iC$ the subcategory of isomorphisms in $C$; equivalently, $iC$ is the maximal subgroupoid in $C$. If $C$ is an algebroid stack then the stack associated to the substack of isomorphisms $\tilde{iC}$ is a gerbe.

For an algebra $K$ we denote by $K^+$ the linear category with a single object whose endomorphism algebra is $K$. For a sheaf of algebras $\mathcal{K}$ on $X$ we denote by $\mathcal{K}^+$ the prestack in linear categories given by $U \mapsto \mathcal{K}(U)^+$. Let $\mathcal{K}^+$ denote the associated stack. Then, $\mathcal{K}^+$ is an algebroid stack equivalent to the stack of locally free $\mathcal{K}^{op}$-modules of rank one.

By a twisted form of $\mathcal{K}$ we mean an algebroid stack locally equivalent to $\mathcal{K}^+$. It is easy to see that the equivalence classes of twisted forms of $\mathcal{K}$ are bijective correspondence with $H^2(X; Z(\mathcal{K})^\times)$, where $Z(\mathcal{K})$ denotes the center of $\mathcal{K}$.

2.5. Twisted forms of $\mathcal{O}_X$. Twisted forms of $\mathcal{O}_X$ are in bijective correspondence with $\mathcal{O}_X^\times$-gerbes: if $\mathcal{S}$ is a twisted form of $\mathcal{O}_X$, the corresponding gerbe is the substack $i\mathcal{S}$ of isomorphisms in $\mathcal{S}$. We shall not make a distinction between the two notions.

The equivalence classes of twisted forms of $\mathcal{O}_X$ are in bijective correspondence with $H^2(X; \mathcal{O}_X^\times)$. The composition

$$\mathcal{O}_X^\times \to \mathcal{O}_X^\times/\mathbb{C}^\times \xrightarrow{\log} \mathcal{O}_X/\mathbb{C} \xrightarrow{\text{DR}} \mathcal{J}_X/\mathcal{O}_X$$

induces the map $H^2(X; \mathcal{O}_X^\times) \to H^2(X; \text{DR}(\mathcal{J}_X/\mathcal{O}_X)) \cong H^2(\Gamma(X; \mathcal{A}_X^\natural \otimes \mathcal{J}_X/\mathcal{O}_X), \nabla^\text{can})$. We denote by $[\mathcal{S}]$ the image in the latter space of the class of $\mathcal{S}$.

The short exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{1} \mathcal{J}_X \to \mathcal{J}_X/\mathcal{O}_X \to 0$$

gives rise to the short exact sequence of complexes

$$0 \to \Gamma(X; \mathcal{A}_X^\natural) \to \Gamma(X; \text{DR}(\mathcal{J}_X)) \to \Gamma(X; \text{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to 0,$$

hence to the map (connecting homomorphism) $H^2(X; \text{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to H^3_{dR}(X)$. Namely, if $B \in \Gamma(X; \mathcal{A}_X^\natural \otimes \mathcal{J}_X)$ maps to $\overline{B} \in \Gamma(X; \mathcal{A}_X^\natural \otimes \mathcal{J}_X/\mathcal{O}_X)$ which represents $[\mathcal{S}]$, then there exists a unique $H \in \Gamma(X; \mathcal{A}^\natural)$ such that $\nabla^\text{can} B = \text{DR}(1)(H)$. The form $H$ is closed and represents the image of the class of $\overline{\mathcal{B}}$ under the connecting homomorphism.

Notation. We denote by $[\mathcal{S}]_{dR}$ the image of $[\mathcal{S}]$ under the map

$$H^2(X; \text{DR}(\mathcal{J}_X/\mathcal{O}_X)) \to H^3_{dR}(X).$$
Deformations of algebroid stacks

3.1. Deformations of linear stacks. Here we describe the notion of 2-groupoid of deformations of an algebroid stack. We follow [2] and refer the reader to that paper for all the proofs and additional details.

For an $R$-linear category $C$ and homomorphism of algebras $R \rightarrow S$ we denote by $C \otimes_R S$ the category with the same objects as $C$ and morphisms defined by $\text{Hom}_{C \otimes_R S}(A, B) = \text{Hom}_C(A, B) \otimes_R S$.

For a prestack $C$ in $R$-linear categories we denote by $C \otimes_R S$ the prestack associated to the fibered category $U \mapsto C(U) \otimes_R S$.

Lemma 3.1 ([2], Lemma 4.13). Suppose that $A$ is a sheaf of $R$-algebras and $C$ is an $R$-algebroid stack. Then $\tilde{C} \otimes_R S$ is an algebroid stack.

Suppose now that $C$ is a stack in $k$-linear categories on $X$ and $R$ is a commutative Artin $k$-algebra. We denote by $\text{Def}(C)(R)$ the 2-category with

- objects: pairs $(B, \varpi)$, where $B$ is a stack in $R$-linear categories flat over $R$ and $\varpi : \tilde{B} \otimes_R k \rightarrow C$ is an equivalence of stacks in $k$-linear categories
- 1-morphisms: a 1-morphism $(B^{(1)}, \varpi^{(1)}) \rightarrow (B^{(2)}, \varpi^{(2)})$ is a pair $(F, \theta)$ where $F : B^{(1)} \rightarrow B^{(2)}$ is a $R$-linear functor and $\theta : \varpi^{(2)} \circ (F \otimes_R k) \rightarrow \varpi^{(1)}$ is an isomorphism of functors
- 2-morphisms: a 2-morphism $(F', \theta') \rightarrow (F'', \theta'')$ is a morphism of $R$-linear functors $\kappa : F' \rightarrow F''$ such that $\theta'' \circ (\text{Id}_{\varpi^{(2)}} \otimes (\kappa \otimes_R k)) = \theta'$

The 2-category $\text{Def}(C)(R)$ is a 2-groupoid.

Let $B$ be a prestack on $X$ in $R$-linear categories. We say that $B$ is flat if for any $U \subseteq X$, $A, B \in B(U)$ the sheaf $\text{Hom}_B(A, B)$ is flat (as a sheaf of $R$-modules).

Lemma 3.2 ([2], Lemma 6.2). Suppose that $B$ is a flat $R$-linear stack on $X$ such that $\tilde{B} \otimes_R k$ is an algebroid stack. Then $B$ is an algebroid stack.

3.2. Deformations of twisted forms of $O$. Suppose that $S$ is a twisted form of $O_X$. We will now describe the DGLA controlling the deformations of $S$.

The complex $\Gamma(X; DR(C^\bullet(J_X))) = (\Gamma(X; A^\bullet_X \otimes C^\bullet(J_X)), \nabla^{can} + \delta)$ is a differential graded brace algebra in a canonical way. The abelian Lie algebra $J_X = C^0(J_X)$ acts on the brace algebra $C^\bullet(J_X)$ by derivations of degree $-1$ by Gerstenhaber bracket. The above action factors through an action of $J_X/O_X$. Therefore, the abelian Lie algebra
Γ(\(X; A^2_X \otimes J_X/O_X\)) acts on the brace algebra \(A^•_X \otimes C^•(J_X)\) by derivations of degree +1. Following longstanding tradition, the action of an element \(a\) is denoted by \(ι_a\).

Due to commutativity of \(J_X\), for any \(ω ∈ Γ(\(X; A^2_X \otimes J_X/O_X\))\) the operation \(ι_ω\) commutes with the Hochschild differential \(δ\). If, moreover, \(ω\) satisfies \(∇^\text{can} ω = 0\), then \(∇^\text{can} + δ + i_ω\) is a square-zero derivation of degree one of the brace structure. We refer to the complex

\[
Γ(X; DR (C^•(J_X)) ± := (Γ(X; A^•_X \otimes C^•(J_X)), \nabla^\text{can} + δ + i_ω)
\]

as the \(ω\)-twist of \(Γ(X; DR (C^•(J_X)))\).

Let \(g_{DR}(J)_ω := Γ(X; DR (C^•(J_X)))[1]_ω\) regarded as a DGLA. The following theorem is proved in [2] (Theorem 1 of loc. cit.):

**Theorem 3.3.** For any Artin algebra \(R\) with maximal ideal \(m_R\) there is an equivalence of 2-groupoids

\[
\text{MC}^2(\mathcal{G}_{DR}(J_X) \otimes m_R) ≅ \text{Def}(S)(R)
\]

natural in \(R\).

### 4. Formality

We give a synopsis of the results of [5] in the notations of loc. cit. Let \(k\) be a field of characteristic zero. For a \(k\)-cooperad \(C\) and a complex of \(k\)-vector spaces \(V\) we denote by \(F^\text{e}_2(V)\) the cofree \(C\)-coalgebra on \(V\).

We denote by \(e_2\) the operad governing Gerstenhaber algebras. The latter is Koszul, and we denote by \(e_2^\vee\) the dual cooperad.

For an associative \(k\)-algebra \(A\) the Hochschild complex \(C^\bullet(A)\) has a canonical structure of a brace algebra, hence a structure of homotopy \(e_2\)-algebra. The latter structure is encoded in a differential (i.e. a coderivation of degree one and square zero) \(M: F^\text{e}_2^\vee(C^\bullet(A)) → F^\text{e}_2^\vee(C^\bullet(A))[1]\).

Suppose from now on that \(A\) is regular commutative algebra over a field of characteristic zero (the regularity assumption is not needed for the constructions). Let \(V^\bullet(A) = \text{Sym}^\bullet(\text{Der}(A)[-1])\) viewed as a complex with trivial differential. In this capacity \(V^\bullet(A)\) has a canonical structure of an \(e_2\)-algebra which gives rise to the differential \(d_{V^\bullet(A)}\) on \(F^\text{e}_2^\vee(V^\bullet(A))\); we have: \(B^\text{e}_2^\vee(V^\bullet(A)) = (F^\text{e}_2^\vee(V^\bullet(A)), d_{V^\bullet(A)})\) (see [5], Theorem 1 for notations).

In addition, the authors introduce a sub-\(e_2^\vee\)-coalgebra \(Ξ(A)\) of both \(F^\text{e}_2^\vee(C^\bullet(A))\) and \(F^\text{e}_2^\vee(V^\bullet(A))\). We denote by \(σ: Ξ(A) → F^\text{e}_2^\vee(C^\bullet(A))\) and \(ι: Ξ(A) → F^\text{e}_2^\vee(V^\bullet(A))\) respective inclusions and identify \(Ξ(A)\)
with its image under the latter one. By [5], Proposition 7 the differential $d_{V^\bullet(A)}$ preserves $\Xi(A)$; we denote by $d_{V^\bullet(A)}$ its restriction to $\Xi(A)$. By Theorem 3, loc. cit. the inclusion $\sigma$ is a morphism of complexes. Hence, we have the following diagram in the category of differential graded $e_2^\vee$-coalgebras:

\[(4.0.1) \quad (\mathbb{F}_{e_2^\vee}(C^\bullet(A)), M) \xrightarrow{\varepsilon} (\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\sigma} (\Xi(A), d_{V^\bullet(A)}) \xrightarrow{i} B_{e_2^\vee}(V^\bullet(A))\]

Applying the functor $\Omega_{e_2}$ (adjoint to the functor $B_{e_2^\vee}$, see [5], Theorem 1) to (4.0.1) we obtain the diagram

\[(4.0.2) \quad \Omega_{e_2}(\mathbb{F}_{e_2^\vee}(C^\bullet(A)), M) \xleftarrow{\Omega_{e_2}(\sigma)} \Omega_{e_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\Omega_{e_2}(i)} \Omega_{e_2}(B_{e_2^\vee}(V^\bullet(A)))\]

of differential graded $e_2$-algebras. Let $\nu = \eta_{e_2} \circ \Omega_{e_2}(i)$, where $\eta_{e_2} : \Omega_{e_2}(B_{e_2^\vee}(V^\bullet(A))) \to V^\bullet(A)$ is the counit of adjunction. Thus, we have the diagram

\[(4.0.3) \quad \Omega_{e_2}(\mathbb{F}_{e_2^\vee}(C^\bullet(A)), M) \xleftarrow{\Omega_{e_2}(\sigma)} \Omega_{e_2}(\Xi(A), d_{V^\bullet(A)}) \xrightarrow{\nu} V^\bullet(A)\]

of differential graded $e_2$-algebras.

**Theorem 4.1** ([5], Theorem 4). The maps $\Omega_{e_2}(\sigma)$ and $\nu$ are quasi-isomorphisms.

Additionally, concerning the DGLA structures relevant to applications to deformation theory, deduced from respective $e_2$-algebra structures we have the following result.

**Theorem 4.2** ([5], Theorem 2). The DGLA $\Omega_{e_2}(\mathbb{F}_{e_2^\vee}(C^\bullet(A)), M)[1]$ and $C^\bullet(A)[1]$ are canonically $L_\infty$-quasi-isomorphic.

**Corollary 4.3** (Formality). The DGLA $C^\bullet(A)[1]$ and $V^\bullet(A)[1]$ are $L_\infty$-quasi-isomorphic.

4.1. **Some (super-)symmetries.** For applications to deformation theory of algebroid stacks we will need certain equivariance properties of the maps described in 4.

For $a \in A$ let $i_a : C^\bullet(A) \to C^\bullet(A)[-1]$ denote the adjoint action (in the sense of the Gerstenhaber bracket and the identification $A = C^0(A)$). It is given by the formula

$$i_a D(a_1, \ldots, a_n) = \sum_{i=0}^n (-1)^k D(a_1, \ldots, a_i, a, a_{k+1}, \ldots, a_n).$$

The operation $i_a$ extends uniquely to a coderivation of $\mathbb{F}_{e_2^\vee}(C^\bullet(A))$; we denote this extension by $i_a$ as well. Furthermore, the subcoalgebra $\Xi(A)$ is preserved by $i_a$. 

Since the operation $i_a$ is a derivation of the cup product as well as of all of the brace operations on $C^\bullet(A)$ and the homotopy-$e_2$-algebra structure on $C^\bullet(A)$ given in terms of the cup product and the brace operations it follows that $i_a$ anti-commutes with the differential $M$. Hence, the coderivation $i_a$ induces a derivation of the differential graded $e_2$-algebra $\Omega_{e_2} (\mathcal{F}_{e_2}(C^\bullet(A)), M)$ which will be denoted by $i_a$ as well. For the same reason the DGLA $\Omega_{e_2} (\mathcal{F}_{e_2}(C^\bullet(A)), M)[1]$ and $C^\bullet(A)[1]$ are quasi-isomorphic in a way which commutes with the respective operations $i_a$.

On the other hand, let $i_a : V^\bullet(A) \rightarrow V^\bullet(A)[-1]$ denote the adjoint action in the sense of the Schouten bracket and the identification $A = V^0(A)$. The operation $i_a$ extends uniquely to a coderivation of the $e_2$-algebra structure on $V^\bullet(A)$. We denote this coderivation as well as its unique extension to a derivation of the differential graded $e_2$-algebra $\Omega_{e_2} (\mathcal{B}_{e_2}(V^\bullet(A)))$ by $i_a$.

The subcoalgebra $\Xi(A)$ of $\mathcal{F}_{e_2}(C^\bullet(A))$ and $\mathcal{F}_{e_2}(V^\bullet(A))$ is preserved by the respective operations $i_a$. Moreover, the restrictions of the two operations to $\Xi(A)$ coincide, i.e. the maps in (4.0.1) commute with $i_a$ and, therefore, so do the maps in (4.0.2) and (4.0.3).

4.2. Deformations of $\mathcal{O}$ and Kontsevich formality. Suppose that $X$ is a manifold. Let $\mathcal{O}_X$ (respectively, $\mathcal{I}_X$) denote the structure sheaf (respectively, the sheaf of vector fields). The construction of the diagram localizes on $X$ yielding the diagram of sheaves of differential graded $e_2$-algebras

\[
\begin{align*}
\Omega_{e_2}(\mathcal{F}_{e_2}(C^\bullet(\mathcal{O}_X)), M) & \xrightarrow{\Omega_{e_2}(\sigma)} \Omega_{e_2}(\Xi(\mathcal{O}_X), d_{V^\bullet(\mathcal{O}_X)}) \\
& \xrightarrow{\nu} V^\bullet(\mathcal{O}_X),
\end{align*}
\]

where $C^\bullet(\mathcal{O}_X)$ denotes the sheaf of multidifferential operators and $V^\bullet(\mathcal{O}_X) := \text{Sym}^\bullet_{\mathcal{O}_X}(\mathcal{I}_X[-1])$ denotes the sheaf of multivector fields. Theorem 4.1 extends easily to this case stating that the morphisms $\Omega_{e_2}(\sigma)$ and $\nu$ in (4.2.1) are quasi-isomorphisms of sheaves of differential graded $e_2$-algebras.

5. Formality for the algebroid Hochschild complex

5.1. A version of [5] for jets. Let $C^\bullet(\mathcal{J}_X)$ denote sheaf of continuous (with respect to the adic topology) $\mathcal{O}_X$-multilinear Hochschild cochains on $\mathcal{J}_X$. Let $V^\bullet(\mathcal{J}_X) = \text{Sym}^\bullet_{\mathcal{J}_X}(\text{Der}_{\mathcal{O}_X}^{\text{cont}}(\mathcal{J}_X)[-1])$. 

Working now in the category of graded \( \mathcal{O}_X \)-modules we have the diagram
\[(5.1.1) \quad \Omega_{e_2}(F_{e_2}(C^*(J_X)), M) \xleftarrow{\Omega_{e_2}(\sigma)} \Omega_{e_2}(\Xi(J_X), d_{V^*}(J_X)) \xrightarrow{\nu} V^*(J_X)\]
of sheaves of differential graded \( \mathcal{O}_X \)-e_2-algebras. Theorem 4.1 extends easily to this situation: the morphisms \( \Omega_{e_2}(\sigma) \) and \( \nu \) in (5.1.1) are quasi-isomorphisms. The sheaves of DGLA \( \Omega_{e_2}(F_{e_2}(C^*(J_X)), M)[1] \) and \( C^*(J_X)[1] \) are canonically \( L_\infty \)-quasi-isomorphic.

The canonical flat connection \( \nabla^{can} \) on \( J_X \) induces a flat connection which we denote by \( \nabla^{can} \) as well on each of the objects in the diagram (5.1.1). Moreover, the maps \( \Omega_{e_2}(\sigma) \) and \( \nu \) are flat with respect to \( \nabla^{can} \)

due to the maps of respective de Rham complexes
\[(5.1.2) \quad \text{DR}(\Omega_{e_2}(F_{e_2}(C^*(J_X)), M)) \xleftarrow{\text{DR}(\Omega_{e_2}(\sigma))} \text{DR}(\Omega_{e_2}(\Xi(J_X), d_{V^*}(J_X))) \xrightarrow{\text{DR}(\nu)} \text{DR}(V^*(J_X)) \]
where, for \( (K^*, d) \) one of the objects in (5.1.1) we denote by \( \text{DR}(K^*, d) \) the total complex of the double complex \( (A_X^* \otimes K^*, d, \nabla^{can}) \). All objects in the diagram (5.1.2) have canonical structures of differential graded e_2-algebras and the maps are morphisms thereof.

The DGLA \( \Omega_{e_2}(F_{e_2}(C^*(J_X)), M)[1] \) and \( C^*(J_X)[1] \) are canonically \( L_\infty \)-quasi-isomorphic in a way compatible with \( \nabla^{can} \). Hence, the DGLA \( \text{DR}(\Omega_{e_2}(F_{e_2}(C^*(J_X)), M)[1]) \) and \( \text{DR}(C^*(J_X)[1]) \) are canonically \( L_\infty \)-quasi-isomorphic.

5.2. A version of [5] for jets with a twist. Suppose that \( \omega \in \Gamma(X; A_X^* \otimes J_X/\mathcal{O}_X) \) satisfies \( \nabla^{can}\omega = 0 \).

For each of the objects in (5.1.2) we denote by \( i_\omega \) the operation which is induced by the one described in 4.1 and the wedge product on \( A_X^* \).
Thus, for each differential graded e_2-algebra \( (N^*, d) \) in (5.1.2) we have a derivation of degree one and square zero \( i_\omega \) which anticommutes with \( d \) and we denote by \( (N^*, d, \omega) \) the \( \omega \)-twist of \( (N^*, d) \), i.e. the differential graded e_2-algebra \( (N^*, d + i_\omega) \). Since the morphisms in (5.1.2) commute with the respective operations \( i_\omega \), they give rise to morphisms of respective \( \omega \)-twists
\[(5.2.1) \quad \text{DR}(\Omega_{e_2}(F_{e_2}(C^*(J_X)), M))_\omega \xleftarrow{\text{DR}(\Omega_{e_2}(\sigma))} \text{DR}(\Omega_{e_2}(\Xi(J_X), d_{V^*}(J_X)))_\omega \xrightarrow{\text{DR}(\nu)} \text{DR}(V^*(J_X))_\omega \]

Let \( F_*A_X^* \) denote the stupid filtration: \( F_iA_X^* = A_X^{\geq -i} \). The filtration \( F_*A_X^* \) induces a filtration denoted \( F_i\text{DR}(K^*, d)_\omega \) for each object \( (K^*, d) \).
of (5.1.1) defined by \( F_i \mathcal{D}R(K^\bullet, d) = F_i \mathcal{A}_X^\bullet \otimes K^\bullet \). As is easy to see, the associated graded complex is given by
\[
Gr_{−p} \mathcal{D}R(K^\bullet, d)_\omega = (\mathcal{A}_X^p \otimes K^\bullet, \text{Id} \otimes d).
\]

It is clear that the morphisms \( \mathcal{D}R(\Omega_{e_2}(\sigma)) \) and \( \mathcal{D}R(\nu) \) are filtered with respect to \( F_i \).

**Theorem 5.1.** The morphisms in (5.2.1) are filtered quasi-isomorphisms, i.e. the maps \( Gr_{−p} \mathcal{D}R(\Omega_{e_2}(\sigma)) \) and \( Gr_{−p} \mathcal{D}R(\nu) \) are quasi-isomorphisms for all \( i \in \mathbb{Z} \).

**Proof.** We consider the case of \( \mathcal{D}R(\Omega_{e_2}(\sigma)) \) leaving \( Gr_{−p} \mathcal{D}R(\nu) \) to the reader.

The map \( Gr_{−p} \mathcal{D}R(\Omega_{e_2}(\sigma)) \) induced by \( \mathcal{D}R(\Omega_{e_2}(\sigma)) \) on the respective associated graded objects in degree \( −p \) is equal to the map of complexes
\[
\text{Id} \otimes \Omega_{e_2}(\sigma): \mathcal{A}_X^p \otimes \Omega_{e_2}(\ell_i(\mathcal{J}_X), d_{i^*}(\mathcal{J}_X)) \to \mathcal{A}_X^p \otimes \Omega_{e_2}(C^*(\mathcal{J}_X)), M).
\]

The map \( \sigma \) is a quasi-isomorphism by Theorem 4.1, therefore so is \( \Omega_{e_2}(\sigma) \). Since \( \mathcal{A}_X^p \) is flat over \( \mathcal{O}_X \), the map (5.2.3) is a quasi-isomorphism. \( \square \)

**Corollary 5.2.** The maps \( \mathcal{D}R(\Omega_{e_2}(\sigma)) \) and \( \mathcal{D}R(\nu) \) in (5.2.1) are quasi-isomorphisms of sheaves of differential graded \( e_2 \)-algebras.

Additionally, the DGLA \( \mathcal{D}R(\Omega_{e_2}(\mathbb{F}_{e_2}(C^*(\mathcal{J}_X)), M)[1]) \) and \( \mathcal{D}R(C^*(\mathcal{J}_X))[1] \) are canonically \( L_\infty \)-quasi-isomorphic in a way which commutes with the respective operations \( i_\omega \) which implies that the respective \( \omega \)-twists \( \mathcal{D}R(\Omega_{e_2}(\mathbb{F}_{e_2}(C^*(\mathcal{J}_X)), M)[1])_\omega \) and \( \mathcal{D}R(C^*(\mathcal{J}_X))[1])_\omega \) are canonically \( L_\infty \)-quasi-isomorphic.

5.3. \( L_\infty \)-structures on multivectors. The canonical pairing \( \langle \ , \rangle : \mathcal{A}_X^1 \otimes \mathcal{T}_X \to \mathcal{O}_X \) extends to the pairing
\[
\langle \ , \rangle : \mathcal{A}_X^1 \otimes V^\bullet(\mathcal{O}_X) \to V^\bullet(\mathcal{O}_X)[−1]
\]

For \( k \geq 1, \omega = \alpha_1 \wedge \ldots \wedge \alpha_k, \alpha_i \in \mathcal{A}_X^1, i = 1, \ldots, k \), let
\[
\Phi(\omega): \text{Sym}^k V^\bullet(\mathcal{O}_X)[2] \to V^\bullet(\mathcal{O}_X)[k]
\]
denote the map given by the formula
\[
\Phi(\omega)(\pi_1, \ldots, \pi_k) = (-1)^{(k−1)|\pi_1|−1+\ldots+2|\pi_{k−3}|−1+|\pi_{k−2}|−1)} \sum_{\sigma} \text{sgn}(\sigma) \langle \alpha_1, \pi_{\sigma(1)} \rangle \wedge \cdots \wedge \langle \alpha_k, \pi_{\sigma(k)} \rangle,
\]

where \( |\pi| = l \) for \( \pi \in V^l(\mathcal{O}_X) \). For \( \alpha \in \mathcal{O}_X \) let \( \Phi(\alpha) = \alpha \in V^0(\mathcal{O}_X) \).
Recall that a graded vector space $W$ gives rise to the graded Lie algebra $\text{Der}(\text{coComm}(W[1]))$. An element $\gamma \in \text{Der}(\text{coComm}(W[1]))$ of degree one which satisfies $[\gamma, \gamma] = 0$ defines a structure of an $L_\infty$-algebra on $W$. Such a $\gamma$ determines a differential $\partial_\gamma := [\gamma, \cdot]$ on $\text{Der}(\text{coComm}(W[1]))$, such that $(\text{Der}(\text{coComm}(W[1])), \partial_\gamma)$ is a differential graded Lie algebra. If $g$ is a graded Lie algebra and $\gamma$ is the element of $\text{Der}(\text{coComm}(g[1]))$ corresponding to the bracket on $g$, then $(\text{Der}(\text{coComm}(g[1])), \partial_\gamma)$ is equal to the shifted Chevalley cochain complex $C^\bullet(g; g)[1]$.

In what follows we consider the (shifted) de Rham complex $A^\bullet_X[2]$ as a differential graded Lie algebra with the trivial bracket.

**Lemma 5.3.** The map $\omega \mapsto \Phi(\omega)$ defines a morphism of sheaves of differential graded Lie algebras

\[(5.3.1) \quad \Phi: A^\bullet_X[2] \to C^\bullet(V^\bullet(O_X)[1]; V^\bullet(O_X)[1])[1].\]

**Proof.** Recall the explicit formulas for the Schouten bracket. If $f$ and $g$ are functions and $X_1, Y_1$ are vector fields, then

\[
[fX_1 \ldots X_k, gY_1 \ldots Y_l] = \sum_i (-1)^{k-i} fX_k(g)X_1 \ldots \widehat{X}_i \ldots X_k Y_1 \ldots Y_l + \\
\sum_j (-1)^j gY_j(f)X_1 \ldots X_k Y_1 \ldots \widehat{Y}_j \ldots Y_l + \\
\sum_{i,j} (-1)^{i+j} f gX_1 \ldots \widehat{X}_i \ldots X_k Y_1 \ldots \widehat{Y}_j \ldots Y_l
\]

Note that for a one-form $\omega$ and for vector fields $X$ and $Y$

\[(5.3.2) \quad \langle \omega, [X, Y] \rangle - \langle [\omega, X], Y \rangle - \langle X, [\omega, Y] \rangle = \Phi(d\omega)(X, Y)\]

From the two formulas above we deduce by an explicit computation that

\[
\langle \omega, [\pi, \rho] \rangle - \langle [\omega, \pi], \rho \rangle - (-1)^{|\pi|-1} \langle \pi, [\omega, \rho] \rangle = (-1)^{|\pi|-1} \Phi(d\omega)(\pi, \rho)
\]

Note that Lie algebra cochains are invariant under the symmetric group acting by permutations multiplied by signs that are computed by the following rule: a permutation of $\pi_i$ and $\pi_j$ contributes a factor $(-1)^{|\pi_i||\pi_j|}$. We use the explicit formula for the bracket on the Lie algebra complex.

\[
[\Phi, \Psi] = \Phi \circ \Psi - (-1)^{|\Phi||\Psi|} \Psi \circ \Phi
\]

\[(\Phi \circ \Psi)(\pi_1, \ldots, \pi_{k+l-1}) = \sum_{I, J} \epsilon(I, J) \Phi(\Psi(\pi_{j_1}, \ldots, \pi_{i_k}), \pi_{j_1}, \ldots, \pi_{j_{l-1}})\]
Here \( I = \{ i_1, \ldots, i_k \}; \; J = \{ j_1, \ldots, j_{l-1} \}; \; i_1 < \ldots < i_k; \; j_1 < \ldots < j_{l-1}; \; I \coprod J = \{ 1, \ldots, k + l - 1 \}; \) the sign \( \epsilon (I, J) \) is computed by the same sign rule as above. The differential is given by the formula

\[
\partial \Phi = [m, \Phi]
\]

where \( m(\pi, \rho) = (-1)^{|\pi|-1}[\pi, \rho] \). Let \( \alpha = \alpha_1 \ldots \alpha_k \) and \( \beta = \beta_1 \ldots \beta_l \). We see from the above that both cochains \( \Phi(\alpha) \circ \Phi(\beta) \) and \( \Phi(\beta) \circ \Phi(\alpha) \) are antisymmetrizations with respect to \( \alpha \) and \( \beta \).

\[
\sum_{I,J,p} \pm \langle \alpha_1 \beta_1, \pi_p \rangle \langle \alpha_2, \pi_{i_1} \rangle \cdots \langle \alpha_k, \pi_{i_{k-1}} \rangle \langle \beta_2, \pi_{j_1} \rangle \cdots \langle \beta_l, \pi_{j_{l-1}} \rangle
\]

over all partitions \( \{ 1, \ldots, k + l - 1 \} = I \coprod J \coprod \{ p \} \) where \( i_1 < \ldots < i_{k-1} \) and \( j_1 < \ldots < j_{l-1} \); hence \( \langle \alpha \beta, \pi \rangle = \langle \alpha, \beta, \pi \rangle \). After checking the signs, we conclude that \( [\Phi(\alpha), \Phi(\beta)] = 0 \). Also, from the definition of the differential, we see that \( \partial \Phi(\alpha)(\pi_1, \ldots, \pi_{k+1}) \) is the antisymmetrizations with respect to \( \alpha \) and \( \beta \) of the sum

\[
\sum_{i<j} \pm (\langle \alpha_1, [\pi_i, \pi_j] \rangle - \langle [\alpha_1, \pi_i], \pi_j \rangle - (-1)^{|\pi_i|-1}[\pi_i, \langle \alpha_1, \pi_j \rangle])
\]

\[
= \langle \alpha_2, \pi_1 \rangle \cdots \langle \alpha_i, \pi_{i-1} \rangle \langle \alpha_{i+1}, \pi_{i+1} \rangle \cdots \langle \alpha_{j-1}, \pi_{j-1} \rangle \langle \alpha_j, \pi_{j+1} \rangle \langle \alpha_k, \pi_{k+1} \rangle
\]

We conclude from this and (5.3.2) that \( \partial \Phi(\alpha) = \Phi(\delta \alpha) \). \( \square \)

Thus, according to Lemma 5.3, a closed 3-form \( H \) on \( X \) gives rise to a Maurer-Cartan element \( \Phi(H) \) in \( \Gamma(X; C^\bullet(V^\bullet(O_X)[1]; V^\bullet(O_X)[1])[1]) \), hence a structure of an \( L^\infty \)-algebra on \( V^\bullet(O_X)[1] \) which has the trivial differential (the unary operation), the binary operation equal to the Schouten-Nijenhuis bracket, the ternary operation given by \( \Phi(H) \), and all higher operations equal to zero. Moreover, cohomologous closed 3-forms give rise to gauge equivalent Maurer-Cartan elements, hence to \( L^\infty \)-isomorphic \( L^\infty \)-structures.

**Notation.** For a closed 3-form \( H \) on \( X \) we denote the corresponding \( L^\infty \)-algebra structure on \( V^\bullet(O_X)[1] \) by \( V^\bullet(O_X)[1]_H \). Let

\[
s(O_X)_H := \Gamma(X; V^\bullet(O_X)[1])_H.
\]

5.4. \( L^\infty \)-structures on multivectors via formal geometry. In order to relate the results of 5.2 with those of 5.3 we consider the analog of the latter for jets.

Let \( \hat{\Omega}^k_{J/O} := \mathcal{J}X(\mathcal{A}^k_X) \), the sheaf of jets of differential \( k \)-forms on \( X \). Let \( \hat{d}_R \) denote the \( (\mathcal{O}_X\text{-linear}) \) differential in \( \hat{\Omega}^k_{J/O} \) induced by the de Rham differential in \( \mathcal{A}^k_X \). The differential \( \hat{d}_R \) is horizontal with respect to the canonical flat connection \( \nabla^\text{can} \) on \( \hat{\Omega}^k_{J/O} \), hence we have
the double complex \((\mathcal{A}_X^\bullet \otimes \hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}}, \nabla^{can}, \text{Id} \otimes \hat{d}_R)\) whose total complex is denoted \(\text{DR}(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})\).

Let \(\Phi : \mathcal{O}_X \to \mathcal{J}_X\) denote the unit map (not to be confused with the map \(j^\infty\)); it is an isomorphism onto the kernel of \(\hat{d}_R : \mathcal{J}_X \to \hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}}\) and therefore defines the morphism of complexes \(\Phi : \text{DR}(\mathcal{J}_X) \to \text{DR}(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})\) which is a quasi-isomorphism. The map \(\Phi\) is horizontal with respect to the canonical flat connections on \(\mathcal{O}_X\) and \(\mathcal{J}_X\) (respectively, \(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}}\)), therefore we have the induced map of respective de Rham complexes \(\text{DR}(\Phi) : \mathcal{A}_X^\bullet \to \text{DR}(\mathcal{J}_X)\) (respectively, \(\text{DR}(1) : \mathcal{A}_X^\bullet \to \text{DR}(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})\), a quasi-isomorphism).

Let \(C^\bullet(g(\mathcal{J}_X); g(\mathcal{J}_X))\) denote the complex of continuous \(\mathcal{O}_X\)-multilinear cochains. The map of differential graded Lie algebras

\[
\hat{\Phi} : \hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}}[2] \to C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1]
\]

defined in the same way as (5.3.1) is horizontal with respect to the canonical flat connection \(\nabla^{can}\) and induces the map

\[
\text{DR}(\hat{\Phi}) : \text{DR}(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})[2] \to \text{DR}(C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1])
\]

There is a canonical morphism of sheaves of differential graded Lie algebras

\[
\text{DR}(C^\bullet(V^\bullet(\mathcal{J}_X)[1]; V^\bullet(\mathcal{J}_X)[1])[1]) \to C^\bullet(\text{DR}(V^\bullet(\mathcal{J}_X)[1])); \text{DR}(V^\bullet(\mathcal{J}_X)[1]))[1]
\]

There is a canonical morphism of sheaves of differential graded Lie algebras

\[
\text{DR}(\mathcal{A}_X^\bullet \otimes \mathcal{J}_X) \to \text{DR}(\hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})\]

Therefore, a degree three cocycle in \(\Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X)\) determines an \(L_\infty\)-structure on \(\text{DR}(V^\bullet(\mathcal{J}_X)[1])\) and cohomologous cocycles determine \(L_\infty\)-isomorphic structures.

**Notation.** For a section \(B \in \Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X)\) we denote by \(\overline{B}\) its image in \(\Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X/\mathcal{O}_X)\).

**Lemma 5.4.** If \(B \in \Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X)\) satisfies \(\nabla^{can}\overline{B} = 0\), then

1. \(\hat{d}_R B\) is a (degree three) cocycle in \(\Gamma(X; \hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}})\);
2. there exist a unique \(H \in \Gamma(X; \mathcal{A}_X^3)\) such that \(dH = 0\) and \(\text{DR}(1)(H) = \nabla^{can}B\).

**Proof.** For the first claim it suffices to show that \(\nabla^{can}B = 0\). This follows from the assumption that \(\nabla^{can}\overline{B} = 0\) and the fact that \(\hat{d}_R : \mathcal{A}_X^\bullet \otimes \mathcal{J}_X \to \mathcal{A}_X^\bullet \otimes \hat{\mathcal{O}}_{\mathcal{J}/\mathcal{O}}\) factors through \(\mathcal{A}_X^\bullet \otimes \mathcal{J}_X/\mathcal{O}_X\).

We have: \(\hat{d}_R \nabla^{can}B = \nabla^{can}\hat{d}_R B = 0\). Therefore, \(\nabla^{can}B\) is in the image of \(\text{DR}(1) : \Gamma(X; \mathcal{A}_X^3) \to \Gamma(X; \mathcal{A}_X^3 \otimes \mathcal{J}_X)\) which is injective, whence the existence and uniqueness of \(H\). Since \(\text{DR}(1)\) is a morphism of complexes it follows that \(H\) is closed. \(\square\)
Suppose that $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$ satisfies $\nabla^{can} B = 0$. Then, the differential graded Lie algebra $\text{DR}(\mathfrak{g}(\mathcal{J}_X)/\mathcal{P})$ (the $\mathcal{P}$-twist of $\text{DR}(\mathfrak{g}(\mathcal{J}_X))$) is defined. On the other hand, due to Lemma 5.4, (5.4.2) and (5.4.3), $\hat{\partial}_H B$ gives rise to an $L_\infty$-structure on $\text{DR}(V^\bullet(\mathcal{J}_X)[1])$.

**Lemma 5.5.** The $L_\infty$-structure induced by $\hat{\partial}_H B$ is that of a differential graded Lie algebra equal to $\text{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathcal{P}}$.

**Proof.** Left to the reader. $\square$

**Notation.** For a 3-cocycle $\omega \in \Gamma(X; \text{DR}(\hat{\Omega}^\bullet_{\mathcal{J}/\mathcal{O}}))$ we will denote by $\text{DR}(V^\bullet(\mathcal{J}_X)[1])_\omega$ the $L_\infty$-algebra obtained from $\omega$ via (5.4.2) and (5.4.3). Let

$$s(\mathcal{O}_X)_H := \Gamma(X; \text{DR}(V^\bullet(\mathcal{J}_X)[1])_{\mathcal{P}}).$$

**Remark 5.6.** Lemma 5.5 shows that this notation is unambiguous with reference to the previously introduced notation for the twist. In the notations introduced above, $\hat{\partial}_H B$ is the image of $B$ under the injective map $\Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X/\mathcal{O}_X) \to \Gamma(X; \mathcal{A}_X^2 \otimes \hat{\Omega}^1_{\mathcal{J}/\mathcal{O}})$ which factors $\hat{\partial}_H$ and “allows” us to “identify” $B$ with $\hat{\partial}_H B$.

**Theorem 5.7.** Suppose that $B \in \Gamma(X; \mathcal{A}_X^2 \otimes \mathcal{J}_X)$ satisfies $\nabla^{can} B = 0$. Let $H \in \Gamma(X; \mathcal{A}_X^3)$ denote the unique 3-form such that $\text{DR}(1\langle H \rangle) = \nabla^{can} B$ (cf. Lemma 5.4). Then, the $L_\infty$-algebras $\mathfrak{g}(\mathcal{J}_X)/\mathcal{P}$ and $s(\mathcal{O}_X)_H$ are $L_\infty$-quasi-isomorphic.

**Proof.** The map $j^\infty: V^\bullet(\mathcal{O}_X) \to V^\bullet(\mathcal{J}_X)$ induces a quasi-isomorphism of sheaves of DGLA

$$j^\infty: V^\bullet(\mathcal{O}_X)[1] \to \text{DR}(V^\bullet(\mathcal{J}_X)[1]).$$

Suppose that $H$ is a closed 3-form on $X$. Then, the map (5.4.4) is a quasi-isomorphism of sheaves of $L_\infty$-algebras

$$j^\infty: V^\bullet(\mathcal{O}_X)[1]_H \to \text{DR}(V^\bullet(\mathcal{J}_X)[1]_{\text{DR}(1\langle H \rangle)}).$$

Passing to global section we obtain the quasi-isomorphism of $L_\infty$-algebras

$$j^\infty: s(\mathcal{O}_X)_H \to s(\mathcal{J}_X)_{\text{DR}(1\langle H \rangle)}.$$ (5.4.5)

By assumption, $B$ provides a homology between $\hat{\partial}_H B$ and $\nabla^{can} B = \text{DR}(1\langle H \rangle)$. Therefore, we have the corresponding $L_\infty$-quasi-isomorphism

$$\text{DR}(V^\bullet(\mathcal{J}_X)[1]_{\text{DR}(1\langle H \rangle)}) \cong \text{DR}(V^\bullet(\mathcal{J}_X)[1])_{\partial_H B} = \text{DR}(V^\bullet(\mathcal{J}_X)[1]_{\mathcal{P}})$$

(the second equality is due to Lemma 5.5).
According to Corollary 5.2 the sheaf of DGLA $\mathcal{D}R(V^\bullet(J_X)[1])$ is $L_\infty$-quasi-isomorphic to the DGLA deduced from the differential graded algebra $\mathcal{D}R(\Omega_{e_2}(\mathcal{E}_{e_2}(C^\bullet(J_X)), M))$. The latter DGLA is $L_\infty$-quasi-isomorphic to $\mathcal{D}R(C^\bullet(J_X)[1])$.

Passing to global sections we conclude that $s_{\mathcal{D}R}(J_X)_{\mathcal{D}R}(1)$ and $g_{\mathcal{D}R}(J_X)_{\mathcal{D}R}$ are $L_\infty$-quasi-isomorphic. Together with (5.4.5) this implies the claim.

6. Application to deformation theory

Theorem 6.1. Suppose that $S$ is a twisted form of $\mathcal{O}_X$ (2.5). Let $H$ be a closed 3-form on $X$ which represents $[S]_{dR} \in H^3_{dR}(X)$. For any Artin algebra $R$ with maximal ideal $m_R$ there is an equivalence of 2-groupoids

$$MC^2(s(\mathcal{O}_X)_{H} \otimes m_R) \cong Def(S)(R)$$

natural in $R$.

Proof. Since cohomologous 3-forms give rise to $L_\infty$-quasi-isomorphic $L_\infty$-algebras we may assume, possibly replacing $H$ by another representative of $[S]_{dR}$, that there exists $B \in \Gamma(X; A^2_X \otimes J_X)$ such that $\overline{B}$ represents $[S]$ and $\nabla^{can}B = \mathcal{D}R(1)(H)$. By Theorem 5.7 $s(\mathcal{O}_X)_{H}$ is $L_\infty$-quasi-isomorphic to $g_{\mathcal{D}R}(J_X)_{\mathcal{D}R}$. By the Theorem 2.1 we have a homotopy equivalence of nerves of 2-groupoids $\gamma_{\bullet}(s(\mathcal{O}_X)_{H} \otimes m_R) \cong \gamma_{\bullet}(g_{\mathcal{D}R}(J_X)_{\mathcal{D}R} \otimes m_R)$. Therefore, there are equivalences

$$MC^2(s(\mathcal{O}_X)_{H} \otimes m_R) \cong MC^2(g_{\mathcal{D}R}(J_X)_{\mathcal{D}R} \otimes m_R) \cong Def(S)(R),$$

the second one being that of Theorem 3.3.

Remark 6.2. In particular, the isomorphism classes of formal deformations of $S$ are in a bijective correspondence with equivalence classes of Maurer-Cartan elements of the $L_\infty$-algebra $s_{\mathcal{D}R}(\mathcal{O}_X)_{H} \hat{\otimes} t \cdot \mathbb{C}[t]$. These are the formal twisted Poisson structures in the terminology of [13], i.e. the formal series $\pi = \sum_{k=1}^{\infty} i^k \pi_k$, $\pi_k \in \Gamma(X; \wedge^2 T_X)$, satisfying the equation

$$[\pi, \pi] = \Phi(H)(\pi, \pi, \pi).$$

A construction of an algebroid stack associated to a twisted Poisson structure was proposed by P. Ševera in [12].

References


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