Brief Solution Descriptions and Technique Summary

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1 Techniques

Here is a summary of some techniques. It is by no means an exhaustive list.

Whenever you need a subgroup, a normal subgroup, or a homomorphism, consider group actions. You may not need them, but they often make life much simpler for you. It is a fantastic idea to familiarize yourself with the following standard group actions:
1)G acts on itself by left multiplication
2)G acts on a subgroup H or itself by conjugation
3)G acts on the set of its subsets by conjugation
4)G acts on the set of left cosets of a subgroup H by multiplication
5)G acts on the set of p-Sylow subgroups by conjugation
There are many other group actions that come up in the course of doing problems, but these 5 seem to be the most common.

The orbit-stabiliser theorem - Given an orbit O containing the element x, we have $|O| = [G : G_x]$. G_x is called the stabiliser of x. Its definition is $G_x = \{g \in G | gx = x\}$. One could try to directly calculate the size of an orbit, but for many actions this would be quite hard. Thus, if you want to know the size of an orbit, you should always try the orbit-stabiliser theorem first. Also, the fact that $G_x \leq G$ limits the possibilities for the size of the orbit by Lagrange's theorem. Note that sometimes it is not obvious that we wanted to know the size of an orbit of some group action (e.g. Jan90 #1 doesn't explicitly tell you this).

Note that familiarity with product groups (groups of the form HK) is essential in both non-Sylow problems and Sylow problems. The main theorems to know are the formula $|HK| = \frac{|H||K|}{|H| \cap |K|}$, the fact that $K \trianglelefteq G$ implies HK is a group (actually a weaker hypothesis suffices - see p95), and the fact that $K \trianglelefteq G$ and $H \trianglelefteq G$ implies $HK \trianglelefteq G$. The last one is particularly useful for classification problems because in those problems you want to obtain large normal subgroups of G.

One can classify groups without directly using semidirect products. However, one nice thing about semidirect products is the fact that if you have $H \trianglelefteq G$, $HK = G$ and $H \cap K = \{1\}$, then $G \cong H \rtimes K$. So, you can just quote this theorem and go about figuring out what the possible semidirect products of H and K are. Without this theorem, you might have to argue why the groups you listed are the ONLY isomorphism types. One unfortunate thing that can occur is that two different homomorphisms from K into Aut(H) might give rise to isomorphic semidirect products. So, you have to figure out which ones are isomorphic, if any. For this it helps to be really competent with how things multiply within the semidirect product.

The correspondence theorem (aka the fourth isomorphism theorem) is useful in many contexts. One common theme is to use information in some quotient group and pull it back to the original group. For example, two of the Sylow problems could be done by finding normal subgroups in the quotient group (using Sylow III) to obtain normal subgroups in the original group. The power of the correspondence theorem theme (and the group action theme) becomes particularly clear when you go through proofs of the Sylow theorems and proofs about p-groups.

2 Quick Descriptions of Problem Solutions

Note that there are ways other than what I give here to do these problems.

2.1 Non-Sylow Group Problems

Spr79 #1 - Suppose $\sigma \in Z(Aut(G))$. Show that $\sigma(g) = g \ \forall g \in G$. The main trick is to consider what conjugating a conjugation automorphism by σ does. After that everything unwinds nicely. Aside from keeping the inner automorphisms (the ones that are conjugation) in mind when doing problems involving Aut(G), I'm not sure what the moral of this problem is.

Spr82 #1 - Counting argument (pairing g with g^{-1} among the non-order 2 elements) yields part a. Part b you obtain a contradiction by embedding a 4-element group into G.

Jan86 #1 - Use the formula $|HK| = \frac{|H||K|}{|H|\cap|K|}$ with $H = A_n$ and $K = C_{S_n}(\alpha)$. Note that $H \cap K = C_{A_n}(\alpha)$. Now note that $HK = A_n$ or S_n. Finish the problem off with the orbit-stabiliser theorem (see below for statement of theorem) and some simple algebra.

Jan89 #1 - Let G act on the set of left cosets of H by left multiplication. This action induces a homomorphism $\phi : G \to S_{[G:H]}$. Ker $\phi \trianglelefteq G$. A calculation shows that Ker $\phi \leq H$ and the 1st isomorphism theorem yields [G:Ker ϕ] $\leq$ n!

Aug89 #3i - $g^n H = H$ for some $n \in Z$. Thus $g^n \in H$, so $(g^n)^m = e$ for some $m \in Z$.

Jan90 #1 - If G acts on P(G) (the set of subsets of G) by conjugation, then the set of conjugates of H is the orbit of H under this action. Apply the orbit-stabiliser theorem to find that $[G : N_G(H)]$ is the number of conjugates of H. $|H| \leq |N_G(H)|$ is then used to obtain a contradiction.

Jan91 #1a - Unwind the definitions and whatnot.

Aug93 #1 - If $K \trianglelefteq G$ and $H \leq G$ then $H \vee K = HK$. Apply this theorem to the subgroups in the problem and manipulate things elementwise.
Jan96 #6 - Use group actions and the orbit-stabiliser theorem. Depending on your choice of group action you may also need to exhibit a bijection between the stabiliser and $x^{-1}Hx \cap K$.

Jan99 #2 - Unsure how to briefly describe this one.

2.2 Sylow Problems

Fal84 #2 - G is simple so there are 8 7-Sylow subgroups by Sylow III. Sylow II says that conjugation on the set of 7-Sylow subgroups is a group action. This induces a homomorphism $\phi : G \to S_8$. Ker $\phi = \{1\}$ otherwise we contradict simplicity. The result then follows from the 1st isomorphism theorem.

Jan90 #2 - Sylow III gives us that the 11-Sylow subgroup P and the 13-Sylow subgroup Q are normal in G. Thus $PQ \trianglelefteq G$. There is a 2-Sylow subgroup R of order 2 by Sylow I. So, PQR = G and $PQ \cap R = \{1\}$ by Lagrange. Thus $G \cong PQ \rtimes R$. Now determine the possible homomorphisms $\phi : R \to Aut(PQ)$ to determine the possible structures for the semidirect product of PQ and R. Check whether any of the distinct homomorphisms give rise to isomorphic groups.

Sum91 #5 - Look at G/K. Sylow III tells us there is a 25 element normal subgroup of G/K. The correspondence theorem (Dummit + Foote p100 Theorem 20) tells us there is normal subgroup of G of order 2500.

Jan95 #6 - The simplicity of A_5 and the fact that 5-cycles are even permutations combine to eliminate the possibility that $n_5 = 1$. After that, numerical arguments eliminate various possibilities for n_2 and n_3. For part b, note that we can embed D_8 into S_5. Since all the 2-Sylow subgroups are isomorphic (by Sylow II), they must all be D_8. They act on 5 symbols by fixing one and then treating the other 4 like a square. The 3-Sylow subgroups are 3-cycles, so they fix 2 symbols and rotate the other 3.

Aug96 #6 - Use the direct product recognition theorem (Theorem 9 p173) to show that G is a direct product of its p-Sylow subgroups. This tells us that G is nilpotent, hence solvable.

Sum97 #1 - Sylow III gives us that the 11-Sylow subgroup P and the 13-Sylow subgroup Q are normal in G. Thus $PQ \trianglelefteq G$. There is a 3-Sylow subgroup R of order 2 by Sylow I. So, PQR = G and $PQ \cap R = \{1\}$ by Lagrange. Now apply the semidirect product recognition theorem (Theorem 12 p182). This gives $G \cong PQ \rtimes R$. Now determine the possible homomorphisms $\phi : R \to Aut(PQ)$ to determine the possible structures for the semidirect product of PQ and R. Check whether any of the distinct homomorphisms give rise to isomorphic groups. Unlike Jan90 #2, distinct homomorphisms give rise to isomorphic groups.

Aug00 #2 - Use Sylow III to obtain a normal subgroup N of order 5^3. Then look at G/N. Sylow III tells us G/N has normal subgroups of order 7 and 17. The correspondence theorem then yields normal subgroups of G of order $5^3 * 17$ and $5^3 * 7$.

Jan02 #1 - Not sure how to describe this but the formula $|HK| = \frac{|H||K|}{|H|\cap|K|}$ plays a big role in part a.

Jan02 #2 - This is a semidirect product problem, but it guides you through it. For part d you can determine that the order 7 and order 5 groups are normal from the multiplication. Then use Sylow III to get the number of 3-Sylow subgroups.

Jan04 #2 - Part 1 you use Sylow III. Part 2 use the fact that G/Z(G) is cyclic $\Rightarrow$ G is abelian. Part 3 you can use the class equation combined with the orbit-stabiliser theorem or Burnside's formula.