Midterm 1

Linear Algebra: Matrix Methods

MATH 2130

Fall 2022

Friday September 23, 2022

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	10	30	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 45 minutes to complete the exam.

1. (20 points) • *Find all solutions to the following system of linear equations:*

$$\begin{aligned} 3x_1 &+ 9x_2 &+ 27x_3 &= -3 \\ -3x_1 &- 11x_2 &- 35x_3 &= 5 \\ 2x_1 &+ 8x_2 &+ 26x_3 &= -4 \end{aligned}$$

SOLUTION:

Solution. The solutions to the system of linear equations are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

or equivalently (substituting $t = -x_3$): x_3 is free, $x_2 = -4x_3 - 1$, and $x_1 = 3x_3 + 2$.

To find this, we row reduce the associated augmented matrix

$$\left[\begin{array}{ccc|c} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right]$$

$$\begin{array}{c} R_1' = \frac{1}{3}R_1 \\ \\ R_3' = \frac{1}{2}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 9 & -1 \\ -3 & -11 & -35 & 5 \\ 1 & 4 & 13 & -2 \end{array}\right]$$

$$\begin{array}{c} R_2' = 3R_1 + R_2 \\ R_3' = -R_1 + R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 3 & 9 & -1 \\ 0 & -2 & -8 & 2 \\ 0 & 1 & 4 & -1 \end{array}\right]$$

$$\begin{array}{c} R_2' = R_3 \\ R_3' = R_2' \end{array} \mapsto \quad R_3'' = 2R_2 + R_3 \left[\begin{array}{ccc|c} 1 & 3 & 9 & -1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$R_1' = R_1 - 3R_2 \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Now we modify the RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

Thus the solutions to the system of equations are:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \left\{ \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$$

as claimed. □

1
20 points

2. (10 points) • Consider the linear map ("transformation") $L : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$L(x_1, x_2, x_3) = (2x_1 - x_3, 3x_2 + x_3).$$

Write down the matrix form of ("standard matrix for") the linear map L.

SOLUTION:

Solution. The matrix form of L is

$$\boxed{\begin{bmatrix} 2 & 0 & -1 \\ 0 & 3 & 1 \end{bmatrix}}$$

We find this by computing L on the standard basis elements:

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 1 - 0 \\ 3 \cdot 0 + 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 0 \\ 3 \cdot (1) + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \cdot 0 - 1 \\ 3 \cdot 0 + 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

These give the corresponding columns of the matrix form of L. □

2
10 points

3. • Consider the following matrix A and its corresponding Reduced Row Echelon Form matrix RREF(A):

$$A = \begin{bmatrix} 1 & -3 & 0 & -1 & 4 & -2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 3 & -9 & 0 & -3 & 2 & 4 \\ 1 & -3 & 1 & -2 & 4 & -1 \end{bmatrix} \qquad \text{RREF}(A) = \begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) (3 points) *Are the rows of A linearly independent?*

SOLUTION:

Solution. **No,** there is a zero row in RREF(A). □

(b) (3 points) *Are the columns of A linearly independent?*

SOLUTION:

Solution. **No,** there are columns in RREF(A) that do not have a pivot. □

(c) (3 points) *What is the row rank of A?*

SOLUTION:

Solution. The row rank of A is 3. Indeed, there are 3 non-zero rows in RREF(A). □

(d) (3 points) *What is the column rank of A?*

SOLUTION:

Solution. The column rank of A is 3. Indeed, the row rank and column rank of a matrix are the same. □

(e) (6 points) *Find a basis for the row space of A.*

SOLUTION:

Solution. A basis for the row space of A is given by the non-zero rows of $\mathrm{RREF}(A)$. In other words, a basis for the row space of A is given by

$$\begin{bmatrix} 1 \\ -3 \\ 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

□

(f) (6 points) *Find a basis for the column space of A.*

SOLUTION:

Solution. A basis for the column space of A is given by the columns of A that correspond to the columns in $\mathrm{RREF}(A)$ that have a pivot. In other words, a basis for the column space of A is:

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 2 \\ 4 \end{bmatrix}.$$

□

(g) (6 points) *Find a basis for the space of solutions of the matrix equation* $A\mathbf{x} = \mathbf{0}$.

SOLUTION:

Solution. The modified matrix is

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & 2 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

so a basis for the set of solutions to the matrix equation $A\mathbf{x} = \mathbf{0}$ is

$$\boxed{\begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}.}$$

□

3
30 points

4. • Consider the matrix

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) (10 points) *Find the inverse of B.*

SOLUTION:

Solution. The solution is

$$\boxed{B^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 0 & -1 \\ -3 & -1 & 6 \end{bmatrix}}$$

To do this, we consider the augmented matrix $\left[\begin{array}{c|c} B & I \end{array}\right]$, and do row reduction until we arrive at the matrix $\left[\begin{array}{c|c} I & B^{-1} \end{array}\right]$. In more detail:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & -6 & -1 & -3 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & -6 & -1 & -3 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 & 1 & -6 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & -1 & 6 \end{array}\right]$$

The matrix on the right is the matrix B^{-1}. □

(b) (10 points) *Does there exist* $\mathbf{x} \in \mathbb{R}^3$ *such that* $B\mathbf{x} = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$?

SOLUTION:

Solution. **YES.** Since B is invertible, given any $\mathbf{b} \in \mathbb{R}^3$, we have that $B(B^{-1}\mathbf{b}) = \mathbf{b}$. In particular, for $\mathbf{b} = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$, we have that $\mathbf{x} = B^{-1}\begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$ satisfies $B\mathbf{x} = \begin{bmatrix} 5 \\ \sqrt{2} \\ \pi \end{bmatrix}$.

□

5. (20 points) • The equation

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$

(the *Leontief Production Equation*) arises in the Leontief Input-Output Model. Here $\mathbf{x}, \mathbf{d} \in M_{n\times 1}(\mathbb{R})$ are column vectors and $C \in M_{n\times n}(\mathbb{R})$ is a square matrix. Consider also the equation $\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$ (called the *Price Equation*), where $\mathbf{p}, \mathbf{v} \in M_{n\times 1}(\mathbb{R})$ are column vectors.

Show that

$$\mathbf{p}^T\mathbf{d} = \mathbf{v}^T\mathbf{x}.$$

(This quantity is known as GDP.) [*Hint:* Compute $\mathbf{p}^T\mathbf{x}$ in two ways.]

SOLUTION:

Solution. We are given the equations

$$\mathbf{x} = C\mathbf{x} + \mathbf{d}$$
$$\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$$

Applying $\mathbf{p}^T$ to both sides of the first equation (the Leontief Production Equation) we have

$$\begin{aligned}\mathbf{p}^T\mathbf{x} &= \mathbf{p}^T(C\mathbf{x} + \mathbf{d})\\ &= \mathbf{p}^TC\mathbf{x} + \mathbf{p}^T\mathbf{d}\end{aligned}$$

Now, the transpose of the second equation $\mathbf{p} = C^T\mathbf{p} + \mathbf{v}$ (the Price Equation), is $\mathbf{p}^T = \mathbf{p}^TC + \mathbf{v}^T$, so that applying $\mathbf{x}$ to both sides gives

$$\begin{aligned}\mathbf{p}^T\mathbf{x} &= (\mathbf{p}^TC + \mathbf{v}^T)\mathbf{x}\\ &= \mathbf{p}^TC\mathbf{x} + \mathbf{v}^T\mathbf{x}\end{aligned}$$

Putting the two expressions for $\mathbf{p}^T\mathbf{x}$ together, we have

$$\mathbf{p}^TC\mathbf{x} + \mathbf{p}^T\mathbf{d} = \mathbf{p}^TC\mathbf{x} + \mathbf{v}^T\mathbf{x}$$

Subtracting $\mathbf{p}^TC\mathbf{x}$ from both sides, we arrive at

$$\mathbf{p}^T\mathbf{d} = \mathbf{v}^T\mathbf{x}$$

completing the proof. □

4
20 points