# Solving a system of linear equations with a modified matrix

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# Introduction

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In these notes I explain a technique that makes this slightly faster.

# An example of a system of linear equations

Suppose we want to solve the system of equations:

$$\begin{array}{rcrcrcr} 3x_1 & + & 9x_2 & + & 27x_3 & = & -3 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 5 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & -4 \end{array}$$

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And here is the associated augmented matrix:

$$\left[\begin{array}{rrr|r} 3 & 9 & 27 & -3 \\ -3 & -11 & -35 & 5 \\ 2 & 8 & 26 & -4 \end{array}\right]$$

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$$R_1' = R_1 - 3R_2 \left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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In other words, the matrix we obtain by putting the left hand side of the augmented matrix

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Since the system of equations is consistent (there are no rows that are zero on the left hand side and non-zero on the right), we can try to find all of the solutions to the system of equations.

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- We add such rows until the left hand side of our matrix is a square matrix with only 1 or $-1$ entries on the diagonal.

# Modifying the matrix in our example

Recall that the matrix we obtained by putting the left hand side of the augmented matrix

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The left hand side of the matrix is square, but it does not have only 1 and $-1$ on the diagonal.

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The matrix we obtain is called the **modified matrix**:

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ 0 & 0 & \textcircled{-1} & 0 \end{array}\right]$$

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The matrix we obtain is called the **modified matrix**:

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{\textcircled{-1}} & \color{blue}{0} \end{array}\right]$$

The solutions to our system of equations are determined by certain columns of the modified matrix.

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Assuming the system of equations has a solution (i.e., it is consistent), then the solutions are determined by the last column (green column):

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as well as the columns with the red $-1$ entries (orange column):

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ 0 & 0 & \textcircled{-1} & 0 \end{array}\right]$$

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Having identified the pertinent columns:

$$\left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ 0 & 0 & \textcircled{-1} & 0 \end{array}\right]$$

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the solutions are given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ \textcircled{-1} \end{bmatrix} t + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}$$

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## Proof.

Exercise. □

# Explaining why this works in our example

Recall that in our example, the matrix we obtained by putting the left hand side of our augmented matrix in RREF was:

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This corresponds to the system of equations:

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Clearly $x_3$ is free, $x_2 = -4x_3 - 1$, and $x_1 = 3x_3 + 2$. We can also write this as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad x_3 \in \mathbb{R}$$

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Clearly we go back and forth by setting $t = -x_3$, so both approaches gave the same solutions.

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again we can set $t = -x_3$,

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hopefully giving a sense of *why* the two approaches give the same solutions. The benefit of the latter is that considering our RREF matrix and modified matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{\textcircled{-1}} & \color{blue}{0} \end{array}\right]$$

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hopefully giving a sense of *why* the two approaches give the same solutions. The benefit of the latter is that considering our RREF matrix and modified matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{ccc|c} \textcircled{1} & 0 & -3 & 2 \\ 0 & \textcircled{1} & 4 & -1 \\ \color{blue}{0} & \color{blue}{0} & \color{red}{\textcircled{-1}} & \color{blue}{0} \end{array}\right]$$

we see the vectors in the second solution a little more easily.

# Explaining why this works in our example

Given our solution:

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Clearly there is an easily identified matrix algorithm to give the solution:

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Then the associated augmented matrix is

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The modified matrix is

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and so the solutions are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} t_1 + \begin{bmatrix} 0 \\ -2 \\ 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} t_2 + \begin{bmatrix} 0 \\ -1 \\ 5 \\ 0 \\ 2 \\ -1 \end{bmatrix} t_3 + \begin{bmatrix} 0 \\ 3 \\ 4 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

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We replace $t_1 \mapsto -x_1$, $t_2 \mapsto -x_4$, $t_3 \mapsto -x_6$, and we get

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