Exercise 7.2.23

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 7.2.23 from Lay [LLM16, §7.2]:

Exercise 7.2.23. If λ_1 and λ_2 are the roots of the characteristic polynomial of a 2×2 matrix A, then $p_A(t) = (t - \lambda_1)(t - \lambda_2)$. Use this to show that $\operatorname{tr}(A) = \lambda_1 + \lambda_2$ and $\det(A) = \lambda_1\lambda_2$. (Recall that the trace of a matrix is the sum of the diagonal entries.)

Solution. We will show more generally that for an $n \times n$ matrix A, the trace is the sum of the roots of the characteristic polynomial, and the determinant is the product of the roots. In other words, let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix}$$

and suppose that the characteristic polynomial

$$p_A(t) := \det(tI - A) = \begin{vmatrix} t - a_{11} & \dots & -a_{1n} \\ \vdots & \ddots & \vdots \\ -a_{n1} & \dots & t - a_{nn} \end{vmatrix}$$

has roots $\lambda_1, \dots, \lambda_n$ (where these roots may be complex numbers, in general). Expanding the determinant above in say the first row, one can see that

$$p_A(t) = t^n - (a_{11} + \cdots + a_{nn})t^{n-1} + \cdots .$$

At the same time, setting $t = 0$ in $p_A(t)$ we get the constant term of $p_A(t)$, and from the definition of $p_A(t)$, we have $p_A(0) = \det(0I - A) = \det(-A) = (-1)^n \det(A)$. Putting this together

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with the observation that $a_{11} + \cdots + a_{nn} = \operatorname{tr}(A)$, we see that

$$p_A(t) = t^n - \operatorname{tr}(A)t^{n-1} + \cdots + (-1)^n \det(A). \tag{0.1}$$

On the other hand, if $p_A(t)$ has roots $\lambda_1, \ldots, \lambda_n$, then

$$p_A(t) = (t - \lambda_1) \cdots (t - \lambda_n).$$

Expanding this out, one has

$$p_A(t) = t^n - (\lambda_1 + \cdots + \lambda_n)t^{n-1} + \cdots + (-1)^n \lambda_1 \cdots \lambda_n. \tag{0.2}$$

Putting (0.1) and (0.2) together, we see that

$$\operatorname{tr}(A) = \lambda_1 + \cdots + \lambda_n$$
$$\det(A) = \lambda_1 \cdots \lambda_n.$$

□

References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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