Exercise 6.6.19

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 6.6.19 from Lay [LLM16, §6.6]:

For a least-squares line fitting problem, we have matrices

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}, \ \hat{\boldsymbol{\beta}} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix}$$

where we are given the data of $x_1, \dots, x_m$ and $y_1, \dots, y_m$, and our goal is to find the matrix $\hat{\boldsymbol{\beta}}$, which is a least-squares solution to the matrix equation $X\boldsymbol{\beta} = \mathbf{y}$ (since such a $\hat{\boldsymbol{\beta}}$ will minimize $\|\mathbf{y} - X\hat{\boldsymbol{\beta}}\|^2$, which is the sum of the squares of the errors, $(y_i - (\hat{\beta}_0 + \hat{\beta}x_i))^2$). With the notation above, consider the following numbers:

(i) $\mathrm{SS(R)} := \|X\hat{\boldsymbol{\beta}}\|^2$, the sum of the squares of the "regression" term;
(ii) $\mathrm{SS(T)} := \|\mathbf{y}\|^2$, the sum of the squares for the y-values;
(iii) $\mathrm{SS(E)} := \|\mathbf{y} - X\hat{\boldsymbol{\beta}}\|^2$, the sum of the squares for the "error" term.

Remark 0.1. Although it is not needed for this problem, one can also think of this as follows. If $x_1, \dots, x_m$ are m samples of a random variable x, and $y_1, \dots, y_m$ are m samples of a random variable y, then we have that $\frac{1}{m-1}\mathrm{SS(R)}$ is the sample variance of the random variable $\hat{\beta}_0 + \hat{\beta}_1 x$, that $\frac{1}{m}\mathrm{SS(T)}$ is the sample variance of the random variable y, and that $\frac{1}{m-1}\mathrm{SS(E)}$ is the sample variance of the "error" random variable $y - (\hat{\beta}_0 + \hat{\beta}_1 x)$.

Exercise 6.6.19. Show that $\mathrm{SS(T)} = \mathrm{SS(R)} + \mathrm{SS(E)}$.

Solution. We have that

$$\mathbf{y} = X\hat{\boldsymbol{\beta}} + (\mathbf{y} - X\hat{\boldsymbol{\beta}})$$

Date: November 17, 2022.

Since $X\hat{\boldsymbol{\beta}}$ is the orthogonal projection of $\mathbf{y}$ onto the image of the linear map associated to X, we have $X\hat{\boldsymbol{\beta}} \perp (\mathbf{y} - X\hat{\boldsymbol{\beta}})$ (see for instance the solution to [LLM16, Exe. 6.6.14] for a review of this). From the Pythagorean Theorem (see [LLM16, Thm. 2, p.336]), it follows that

$$\|\mathbf{y}\|^2 = \|X\hat{\boldsymbol{\beta}}\|^2 + \|(\mathbf{y} - X\hat{\boldsymbol{\beta}})\|^2,$$

which by definition is the assertion that $\mathrm{SS(T)} = \mathrm{SS(R)} + \mathrm{SS(E)}$. □

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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