**Exercise 5.5.24**

**Linear Algebra**
**MATH 2130**

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 5.5.24 from Lay [LLM16, §5.5]:

**Exercise 5.5.24.** Let $A$ be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$ and some complex number $\lambda$, then, in fact, $\lambda$ is real and the real part of $\mathbf{x}$ is an eigenvector of $A$.

*Remark* 0.1 (**WARNING**). This exercise is false as stated! As stated, it is not necessarily the case that the real part of $\mathbf{x}$ is an eigenvector of $A$. For instance, let $A$ be the identity matrix, and let $\mathbf{x}$ be any nonzero vector with $\operatorname{Re}\mathbf{x} = 0$. As a concrete example, you can take $A = I$ to be the $2 \times 2$ identity matrix, and $\mathbf{x} = \begin{bmatrix} i \\ i \end{bmatrix}$. Then $A\mathbf{x} = \mathbf{x} = 1 \cdot \mathbf{x}$, but $\operatorname{Re}\mathbf{x} = \mathbf{0}$ is the zero vector, and cannot be an eigenvector for $A$.

The problem should have been written as follows:

**Exercise 5.5.24.** (CORRECTED) Let $A$ be an $n \times n$ real matrix with the property that $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$ and some complex number $\lambda$, then, in fact, $\lambda$ is real. Show moreover that $A(\operatorname{Re}\mathbf{x}) = \lambda \operatorname{Re}\mathbf{x}$ and $A(\operatorname{Im}\mathbf{x}) = \lambda \operatorname{Im}\mathbf{x}$.

*Remark* 0.2. Since at least one of $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ is nonzero (otherwise $\mathbf{x} = \operatorname{Re}\mathbf{x} + i\operatorname{Im}\mathbf{x} = \mathbf{0}$), this means that at least one of $\operatorname{Re}\mathbf{x}$ and $\operatorname{Im}\mathbf{x}$ is a real eigenvector for $\lambda$.

*Solution.* First we will show, more generally, that if $A$ is an $n \times n$ complex matrix with the property $\bar{A}^T A = A$, and $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector $\mathbf{x} \in \mathbb{C}^n$ and some complex number $\lambda$, then, in fact, $\lambda$ is real. To show this, consider that

$$q_A(\mathbf{x}) = \overline{\mathbf{x}}^T A\mathbf{x} = \overline{\mathbf{x}}^T \lambda\mathbf{x} = \lambda\overline{\mathbf{x}}^T\mathbf{x} = \lambda\|\mathbf{x}\|^2 \tag{0.1}$$

*Date*: October 23, 2022.

where $\|\mathbf{x}\|^2 = \overline{\mathbf{x}}^T\mathbf{x} = q_I(\mathbf{x})$ (here $I$ is the $n \times n$ identity matrix). From Exercise 5.5.23, we know that $q_A(\mathbf{x})$ and $q_I(\mathbf{x})$ are real. In fact, if $\mathbf{x} = (x_1, \ldots, x_n)$, then we have

$$\|\mathbf{x}\|^2 = q_I(\mathbf{x}) = \overline{\mathbf{x}}^T\mathbf{x} = \overline{x}_1 x + \cdots + \overline{x}_n x_n = |x_1|^2 + \cdots + |x_n|^2 > 0$$

is positive, since $\mathbf{x} \neq \mathbf{0}$. Therefore, we can divide by $\|\mathbf{x}\|^2$ in (0.1), and we find $\lambda = q_A(\mathbf{x})/\|\mathbf{x}\|^2$ is the quotient of two real numbers, and is therefore real.

Now, assuming that $A$ is real, we will show that $A(\operatorname{Re}\mathbf{x}) = \lambda \operatorname{Re}\mathbf{x}$ and $A(\operatorname{Im}\mathbf{x}) = \lambda \operatorname{Im}\mathbf{x}$. To do this, we will use that $\operatorname{Re}(A\mathbf{x}) = A(\operatorname{Re}\mathbf{x})$ and $\operatorname{Im}(A\mathbf{x}) = A(\operatorname{Im}\mathbf{x})$ (this is asserted on the bottom of [LLM16, p.301], and is given as [LLM16, Exe. 5.5.25, p.303], but we give a proof below). Using this, we see that

$$A(\operatorname{Re}\mathbf{x}) = \operatorname{Re}(A\mathbf{x}) = \operatorname{Re}(\lambda\mathbf{x}) = \lambda \operatorname{Re}\mathbf{x}.$$

$$A(\operatorname{Im}\mathbf{x}) = \operatorname{Im}(A\mathbf{x}) = \operatorname{Im}(\lambda\mathbf{x}) = \lambda \operatorname{Im}\mathbf{x}.$$

This completes the proof. □

Here for completeness we give a proof of the fact that $\operatorname{Re}(A\mathbf{x}) = A(\operatorname{Re}\mathbf{x})$ and $\operatorname{Im}(A\mathbf{x}) = A(\operatorname{Im}\mathbf{x})$. To start, given any complex matrix $Z$, you can check entry-by-entry that:

$$\operatorname{Re} Z = \frac{1}{2}(Z + \overline{Z})$$
$$\operatorname{Im} Z = -\frac{i}{2}(Z - \overline{Z})$$

Then if $B$ is any real matrix of a size so that we can multiply $BZ$, we have

$$\operatorname{Re}(BZ) = \frac{1}{2}(BZ + \overline{BZ}) = \frac{1}{2}(BZ + \overline{B}\,\overline{Z}) = \frac{1}{2}(BZ + B\,\overline{Z}) = \frac{1}{2}(B(Z + \overline{Z}) = B(\frac{1}{2}(Z + \overline{Z})) = B \operatorname{Re} Z.$$

The proof for $\operatorname{Im}(BZ)$ is similar.

## REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, CAMPUS BOX 395, BOULDER, CO 80309

*Email address*: casa@math.colorado.edu