Exercise 5.5.23

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 5.5.23 from Lay [LLM16, §5.5]:

Exercise 5.5.23. Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$, and let $q(\mathbf{x}) = \bar{\mathbf{x}}^T A \mathbf{x}$. Show that $q(\mathbf{x})$ is a real number.

Solution. We will show more generally that if A is an $n \times n$ complex matrix with the property that $\bar{A}^T = A$, and $\mathbf{x}$ is any vector in $\mathbb{C}^n$, then $q(\mathbf{x}) = \bar{\mathbf{x}}^T A \mathbf{x}$ is a real number. To do this, it suffices to show that $\overline{q(\mathbf{x})} = q(\mathbf{x})$.

Indeed, we have

$$
\begin{aligned}
\overline{q(\mathbf{x})} &= \overline{\bar{\mathbf{x}}^T A \mathbf{x} } \\
&= \mathbf{x}^T \overline{A \mathbf{x}} && (\text{see p.300, } \overline{BC} = \overline{B}\,\overline{C}, \text{ and } \overline{\overline{B}} = B) \\
&= \mathbf{x}^T \bar{A} \bar{\mathbf{x}} && (\overline{BC} = \overline{B}\,\overline{C}) \\
&= (\mathbf{x}^T \bar{A} \bar{\mathbf{x}})^T && (\text{transpose of a } 1 \times 1 \text{ matrix}) \\
&= \bar{\mathbf{x}}^T \bar{A}^T (\mathbf{x}^T)^T && (\text{see Thm. 3, p.101, } (BC)^T = C^T B^T) \\
&= \bar{\mathbf{x}}^T A \mathbf{x} && ((B^T)^T = B, \text{ and we assumed } \bar{A}^T = A) \\
&= q(\mathbf{x}).
\end{aligned}
$$

$\square$

REFERENCES

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

UNIVERSITY OF COLORADO, DEPARTMENT OF MATHEMATICS, CAMPUS BOX 395, BOULDER, CO 80309

Email address: casa@math.colorado.edu