Exercise 4.9.18

Linear Algebra
MATH 2130

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ABSTRACT. This is Exercise 4.9.18 from Lay [LLM16, §4.9]:

Exercise 4.9.18. Show that every 2×2 stochastic matrix has at least one steady-state vector. Any such matrix can be written in the form

$$P = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix},$$

where α and β are constants between 0 and 1 (inclusive). (There are two linearly independent steady-state vectors if $\alpha = \beta = 0$. Otherwise there is exactly one.)

Solution. Recall that a steady state vector for a stochastic matrix P is a vector $\mathbf{x}$ with non-negative entries that sum to 1 such that $P\mathbf{x} = \mathbf{x}$. The condition $P\mathbf{x} = \mathbf{x}$ is equivalent to $\mathbf{x} - P\mathbf{x} = \mathbf{0}$, which is equivalent to $(I-P)\mathbf{x} = \mathbf{0}$. Since $I - P = \begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix}$, we have that in matrix form the equation $(I-P)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There are 3 cases we want to consider:

(1) First we assume that $\alpha = \beta = 0$. Then $I - P = 0$, is the zero matrix, so that any vector $\mathbf{x}$ satisfies $(I-P)\mathbf{x} = \mathbf{0}$. In particular the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent steady-state vectors.

(2) Next assume that $\alpha = 0$ and $\beta \neq 0$. Then we have

$$I - P = \begin{bmatrix} 0 & -\beta \\ 0 & \beta \end{bmatrix}$$

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with RREF matrix

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and modified matrix

$$\begin{bmatrix} {\color{red}-1} & {\color{blue}0} \\ 0 & 1 \end{bmatrix}$$

so that a basis for the kernel of $I - P$ is given by

$$\begin{bmatrix} {\color{red}-1} \\ 0 \end{bmatrix}$$

Scaling so that the sum of the entries is 1, we have that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the unique steady state vector for P.

(3) Next assume that $\alpha \neq 0$. Then we have

$$I - P = \begin{bmatrix} \alpha & -\beta \\ -\alpha & \beta \end{bmatrix}$$

with RREF matrix

$$\begin{bmatrix} 1 & -\beta/\alpha \\ 0 & 0 \end{bmatrix}$$

and modified matrix

$$\begin{bmatrix} 1 & -\beta/\alpha \\ {\color{blue}0} & {\color{red}-1} \end{bmatrix}$$

so that a basis for the kernel of $I - P$ is given by

$$\begin{bmatrix} -\beta/\alpha \\ {\color{red}-1} \end{bmatrix}$$

Scaling so that the sum of the entries is 1 (scale the vector by $-\alpha$, then divide by the sum of the entries), we have that $\frac{1}{\alpha+\beta}\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ is the unique steady state vector for P. (Note that even though our derivation required $\alpha \neq 0$, if one were to substitute $\alpha = 0$ into the formula above, one would have the same vector we found in (2).)

□

References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

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