Exercise 4.7.6

Linear Algebra
MATH 2130

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 4.7.6 from Lay [LLM16, §4.7]:

Exercise 4.7.6. Let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ and $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be bases for a vector space V, and suppose that

$$\begin{aligned} \mathbf{f}_1 &= 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3 \\ \mathbf{f}_2 &= 0\mathbf{d}_1 + 3\mathbf{d}_2 + \mathbf{d}_3 \\ \mathbf{f}_3 &= -3\mathbf{d}_1 + 0\mathbf{d}_2 + 2\mathbf{d}_3 \end{aligned} \tag{0.1}$$

a. Find the change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ to the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$.

b. Find the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ for the vector

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3.$$

Solution. a. The change-of-coordinates matrix to go from the coordinates with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ to the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ can be read off from (0.1) as the matrix

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & 3 & 1 \\ -3 & 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}.$$

b. The coordinates for the vector

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$$

with respect to the basis $\mathcal{F} = \{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ are given by

$$(1, -2, 2).$$

Date: October 9, 2022.

Therefore, given part a., the coordinates with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are given by

$$\begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$$

In other words, the coordinates for $\mathbf{x}$ with respect to the basis $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$ are

$$(-4, -7, 3).$$

□

Remark 0.1. Note that in our solution to b., we are claiming that

$$\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3 = -4\mathbf{d}_1 - 7\mathbf{d}_2 + 3\mathbf{d}_3.$$

We could have checked this directly by substituting in the following way:

$$\begin{aligned} \mathbf{x} &= \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3 \\ &= (2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3) - 2(3\mathbf{d}_2 + \mathbf{d}_3) + 2(-3\mathbf{d}_1 + 2\mathbf{d}_3) \\ &= -4\mathbf{d}_1 - 7\mathbf{d}_2 + 3\mathbf{d}_3 \end{aligned}$$

References

[LLM16] David Lay, Stephen Lay, and Judi McDonald, *Linear Algebra and its Applications*, Fifth edition, Pearson, 2016.

University of Colorado, Department of Mathematics, Campus Box 395, Boulder, CO 80309

Email address: casa@math.colorado.edu