Midterm II

Abstract Algebra 1

MATH 3140

Fall 2021

Friday October 29, 2021

NAME: ______________________________

PRACTICE EXAM

SOLUTIONS

Question:	**1**	**2**	**3**	**4**	**5**	Total
Points:	20	20	20	20	20	100
Score:						

- The exam is closed book. You **may not use any resources** whatsoever, other than paper, pencil, and pen, to complete this exam.
- You **may not discuss the exam** with anyone except me, in any way, under any circumstances.
- You **must explain your answers**, and you will be **graded on the clarity of your solutions**.
- Either write your solutions **directly on this exam** or write the solution to **each problem on a separate piece of paper**.
- You must upload your exam as a single **.pdf** to **Canvas**, with the questions in the correct order, etc.
- You have 50 minutes to complete the exam. **Do not forget to leave yourself time (at least 5 minutes) at the end to upload your exam.**

1. (a) (5 points) • *Is the permutation* $\sigma = (1,6,4)(2,5) \in S_6$ *even or odd?*

SOLUTION:

σ is odd.

We have

$$\sigma = (1,6,4)(2,5) = (1,6)(6,4)(2,5)$$

is the product of an odd number of transpositions.

(b) (5 points) *Is the permutation* σ^2 *even or odd?*

SOLUTION:

σ^2 is even.

The square of any permutation is even.

(c) (5 points) *Compute* $|\sigma|$; i.e., the order of the element σ in the group S_6.

SOLUTION:

$|\sigma| = 6$

The order of $(1,6,4)$ is 3 and the order of $(2,5)$ is 2. As σ is equal to the product of these disjoint cycles, it follows that $|\sigma| = \text{lcm}(3,2) = 6$.

(d) (5 points) *With* σ *as above and* $\tau = (5,3,2)$, *compute* $\sigma\tau$ *(as a product of disjoint cycles).*

SOLUTION:

$\sigma\tau = (1,6,4)(3,5)$

We have

$$\sigma\tau = (1,6,4)(2,5)(5,3,2) = (1,6,4)(3,5).$$

1
20 points

2. • Let A be a set, and let $G \leq S_A$ be a subgroup of the group of permutations S_A of A. For an element $a \in A$, define $G_a := \{\sigma \in G : \sigma(a) = a\}$.

(a) (10 points) *For $a \in A$, show that G_a is a subgroup of G.*

SOLUTION

Solution. Certainly we have $e \in G_a$. Now if $\sigma, \tau \in G_a$, then $(\sigma\tau)(a) = \sigma(\tau(a)) = \sigma(a) = a$, so that $\sigma\tau \in G_a$. Finally, if $\sigma \in G_a$, I claim that $\sigma^{-1}(a) = a$, so that $\sigma^{-1} \in G_a$. Indeed, $\sigma(a) = a$, so that applying σ^{-1} to both sides we obtain $\sigma^{-1}(\sigma(a)) = \sigma^{-1}(a)$. Focusing on the left hand side, we have $\sigma^{-1}(\sigma(a)) = (\sigma^{-1}\sigma)(a) = e(a) = a$, proving the claim. Thus G_a is a subgroup. □

(b) (10 points) Let $a, b \in A$, and suppose there exists $\sigma \in G$ such that $b = \sigma(a)$. *Show that G_a and G_b have the same cardinality.*

SOLUTION

Solution. Let $a, b \in A$, and suppose there exists $\sigma \in G$ such that $b = \sigma(a)$. Note that this also implies that $\sigma^{-1}(b) = a$. From the definition of cardinality, we need to show there is a bijective map (or, "one-to-one and onto function") $f : G_a \to G_b$. I claim there is a bijective map (or, "one-to-one and onto function")

$$f : G_a \longrightarrow G_b, \quad \text{given by } \tau \mapsto \sigma\tau\sigma^{-1}.$$

First, let us check this map (or, "function") is well-defined; i.e., that $\sigma\tau\sigma^{-1} \in G_b$. To this end, suppose $\tau \in G_a$. Then $(\sigma\tau\sigma^{-1})(b) = \sigma(\tau(\sigma^{-1}(b)) = \sigma(\tau(a)) = \sigma(a) = b$. Thus $\sigma\tau\sigma^{-1} \in G_b$.

Now let us check that f is bijective (or, "one-to-one and onto") by constructing an inverse map (or, "function")

$$f^{-1} : G_b \longrightarrow G_a, \quad \mu \mapsto \sigma^{-1}\mu\sigma.$$

The same argument as above shows this map (or, "function") is well-defined. Now observe that $f^{-1}f(\tau) = f^{-1}(\sigma\tau\sigma^{-1}) = \sigma^{-1}(\sigma\tau\sigma^{-1})\sigma = \tau$, and $ff^{-1}(\mu) = \sigma(\sigma^{-1}\mu\sigma)\sigma^{-1} = \mu$. Thus f^{-1} is the inverse map (or, "function") of f, and so f is bijective (or, "one-to-one and onto"). Thus, by definition, the cardinality of G_a is the same as the cardinality of G_b. □

2
20 points

3. • Consider the dihedral group D_n, with $n \geq 3$. Recall the notation we have been using: D_n has identity element I, and is generated by elements R and D, satisfying the relations $R^n = D^2 = I$ and $RD = DR^{-1}$. Consider the cyclic subgroup $\langle R^2 \rangle$.

(a) (10 points) *Show that $\langle R^2 \rangle$ is a normal subgroup of D_n.*

SOLUTION

Solution. To show that $\langle R^2 \rangle$ is normal in D_n, it suffices to check for all $g \in D_n$ that $g\langle R^2 \rangle g^{-1} \subseteq \langle R^2 \rangle$. (For a subgroup H of a group G, we have seen that H is normal if and only if $gHg^{-1} \subseteq H$ for all $g \in G$.) So let $R^{a_1}D^{b_1} \in D_n$ and let $R^{2k} \in \langle R^2 \rangle$. Then

$$R^{a_1}D^{b_1}R^{2k}(R^{a_1}D^{b_1})^{-1} = R^{a_1}D^{b_1}R^{2k}D^{b_1}R^{-a_1} = R^{a_1}D^{b_1}D^{b_1}R^{(-1)^{b_1}2k}R^{-a_1} = R^{(-1)^{b_1}2k} \in \langle R^2 \rangle.$$

Thus $\langle R^2 \rangle$ is normal in D_n. □

(b) (10 points) *Find the order of the group $D_n/\langle R^2 \rangle$.* [*Hint:* this may depend on the parity of n.]

SOLUTION

Solution.

$$\boxed{|D_n/\langle R^2 \rangle| = 2 \text{ if } n \text{ is odd, and } 4 \text{ if } n \text{ is even}}$$

To see this, we note that the order of R in D_n is n. Consequently, if n is odd, then $\langle R^2 \rangle = \langle R \rangle$, which has order n. If n is even, then $\langle R^2 \rangle \neq \langle R \rangle$ and the order of $\langle R^2 \rangle$ is $n/2$. By Lagrange's Theorem, the order of $D_4/\langle R^2 \rangle$ is then either $2n/n = 2$ (if n is odd) or $2n/(n/2) = 4$ (if n is even). (Note that in the latter case, the quotient is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, and not to $\mathbb{Z}_4$, since the quotient has two elements of order 2, namely, R and D.) □

3
20 points

4. • Recall that for a commutative ring R with unity $1 \neq 0$, we define $R[x]$ to be the ring of polynomials in x with coefficients in R. Consider the map

$$\phi : \mathbb{Z}[x] \longrightarrow \mathbb{Z}_4[x]$$

$$\sum_{k=0}^{n} a_k x^k \mapsto \sum_{k=0}^{n} [a_k] x^k,$$

where $[a_k] = a_k \pmod 4$.

(a) (10 points) *Show that ϕ is a homomorphism of rings.*

SOLUTION

Solution. First we must show for all $p(x), q(x) \in \mathbb{Z}[x]$ that

$$\phi(p(x) + q(x)) = \phi(p(x)) + \phi(q(x)) \quad \text{and} \quad \phi(p(x)q(x)) = \phi(p(x))\phi(q(x)).$$

To do this, let us suppose that $p(x) = \sum_{k=0}^{n} a_k x^k$ and $q(x) = \sum_{j=0}^{m} b_j x^j$; since addition and multiplication are commutative, we may assume that $n \leq m$, and in fact, taking $a_k = 0$ for $k > n$, we may assume $n = m$. Then

$$\phi(p(x) + q(x)) = \phi\left(\sum_{k=0}^{n} a_k x^k + \sum_{j=0}^{n} b_j x^j\right) = \phi\left(\sum_{k=0}^{n} (a_k + b_k) x^k\right) = \sum_{k=0}^{n} [a_k + b_k] x^k$$

$$= \sum_{k=0}^{n} [a_k] x^k + \sum_{j=0}^{n} [b_j] x^j = \phi(p(x)) + \phi(q(x)).$$

Similarly,

$$\phi(p(x) \cdot q(x)) = \phi\left(\sum_{k=0}^{n} a_k x^k \cdot \sum_{j=0}^{n} b_j x^j\right) = \phi\left(\sum_{i=0}^{2n} \sum_{k=0}^{i} (a_k b_{i-k}) x^i\right) = \sum_{i=0}^{2n} \sum_{k=0}^{i} [a_k][b_{i-k}] x^i$$

$$= \sum_{k=0}^{n} [a_k] x^k \cdot \sum_{j=0}^{n} [b_j] x^j = \phi(p(x)) \cdot \phi(q(x)).$$

Thus ϕ is a homomorphism of rings. □

(b) (10 points) *Describe the kernel of ϕ.* (Do not just write down the definition; you need to describe an explicit subset of $\mathbb{Z}[x]$.)

SOLUTION

Solution. We can describe the kernel as

$$\boxed{\ker \phi = 4\mathbb{Z}[x]}$$

Indeed, suppose that $p(x) = \sum_{k=0}^{n} a_k x^k \in \ker \phi$. Then $[a_k] = 0$ for all $k = 0, \ldots, n$. Thus $a_k \in 4\mathbb{Z}$ for all $k = 0, \ldots, n$. □

4
20 points

5. (20 points) • In a commutative ring with unity, *show that* $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

SOLUTION

Solution. Since we are in a commutative ring with unity, when writing out

$$(a+b)^n = (a+b)(a+b)\cdots(a+b)$$

one can deduce that the number of monomials of the form $a^k b^{n-k}$ in the expansion will be $\binom{n}{k}$, corresponding to choosing k of the n factors above from which to take an a, and then taking a b from the remaining $n-k$ factors.

Here is another argument using induction. First observe that

$$\binom{n}{k-1} + \binom{n}{k} = \frac{n!}{(n-k+1)!(k-1)!} + \frac{n!}{(n-k)!k!} = \frac{n!k}{(n-k+1)!k!} + \frac{n!(n-k+1)}{(n-k+1)!k!}$$

$$= \frac{(n+1)!}{(n+1-k)!k!} = \binom{n+1}{k}.$$

Now, using this, we will prove the assertion of problem using induction. We start with the case $n=1$, and we check that

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = b + a = (a+b)^1.$$

We now perform the inductive step. We assume that $(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}$ for all $m \le n$ for some $n \ge 1$. We then show that

$$(a+b)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

Here is the computation:

$$(a+b)^n(a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}\right)(a+b) = \left(\sum_{k=0}^{n} \binom{n}{k} a^{k+1} b^{n-k}\right) + \left(\sum_{k=0}^{n} \binom{n}{k} a^k b^{n+1-k}\right)$$

$$= \binom{n}{0} b^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k}\right) a^k b^{n+1-k} + \binom{n}{n} a^{n+1}$$

$$= \binom{n+1}{0} b^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^k b^{n+1-k} + \binom{n+1}{n+1} a^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} a^k b^{n+1-k}.$$

□

5
20 points