Exercise 14.30

Abstract Algebra 1
MATH 3140

SEBASTIAN CASALAINA

ABSTRACT. This is Exercise 14.30 from Fraleigh [Fra03, §14]:

Exercise 14.30. Let H be a normal subgroup of a group G, and let $m = (G : H)$. Show that $a^m \in H$ for all $a \in G$.

Solution. Since H is a normal subgroup of G, there is a group structure on G/H, the set of left cosets of H in G, given by the composition rule

$$(aH)(bH) = abH$$

for all $a, b \in G$ ([Fra03, Theorem 14.4 and Corollary 14.5, p.138]). Note that the identity element of G/H is the coset H.

By the definition of the index of a subgroup [Fra03, Definition 10.13, p.101], we have that $|G/H| = (G : H) = m$. From the Corollary of Lagrange's Theorem ([Fra03, Theorem 10.2, p.101]), we know that for any element $aH \in G/H$, the order of aH as an element of the group G/H divides the order of the group G/H, which is m. In other words, for any $a \in G$, we have

$$(aH)^m = H.$$

Using the composition rule, we also get

$$(aH)^m = a^m H.$$

We conclude that

$$a^m H = H$$

and thus that $a^m \in H$. □

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REFERENCES

[Fra03] John Fraleigh, *A First Course in Abstract Algebra*, Seventh edition, Addison Wesley, Pearson, 2003.

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