MIDTERM I
LINEAR ALGEBRA

MATH 2135

Friday February 16, 2018.

Name

PRACTICE EXAM

Please answer the all of the questions, and show your work.
You must explain your answers to get credit.
You will be graded on the clarity of your exposition!

1	2	3	4	5	
20	20	20	20	20	total

Date: February 11, 2018.

1. Give the definition of a vector space.

1
20 points

2
20 points

2. Let V be the $\mathbb{R}$-vector space of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ with addition and scaling given by:

$$\{x_n\}_{n=0}^{\infty} + \{y_n\}_{n=0}^{\infty} = \{x_n + y_n\}_{n=0}^{\infty}, \quad \lambda \cdot \{x_n\}_{n=0}^{\infty} = \{\lambda x_n\}_{n=0}^{\infty},$$

for all $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in V$ and $\lambda \in \mathbb{R}$.

Let $L^1 \subseteq V$ be the subset of sequences $\{x_n\}_{n=0}^{\infty} \in V$ such that $\sum_{n=0}^{\infty} |x_n|$ converges. Let $L^2 \subseteq V$ be the subset of sequences $\{x_n\}_{n=0}^{\infty} \in V$ such that $\sum_{n=0}^{\infty} x_n^2$ converges.

2.(a). Show that L^1 is a subspace of V.

2.(b). Show that L^2 is a subspace of V.

2.(c). Is L^1 contained in L^2?

2.(d). Is L^2 contained in L^1?

3
20 points

3. Let L^2 be the $\mathbb{R}$-vector space consisting of sequences of real numbers $\{x_n\}_{n=0}^{\infty}$ such that $\sum_{n=0}^{\infty} x_n^2$ converges. We have seen there is an inner product on L^2 defined by setting

$$(x, y) = \sum_{n=0}^{\infty} x_n y_n$$

for $x = \{x_n\}_{n=0}^{\infty}, y = \{y_n\}_{n=0}^{\infty} \in L^2$.

Now consider the sequences $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty}$ defined by:

$$x_n = \frac{1}{2^n},$$
$$y_n = \begin{cases} 1 & n = 0, \\ 0 & n > 0. \end{cases}$$

3.(a). Perform the Graeme–Schmidt process to obtain an orthonomal basis for the vector space spanned by $\{x_n\}$ and $\{y_n\}$.

3.(b). Consider the sequence $\{z_n\}_{n=0}^{\infty}$ defined by $z_n = 1/n!$. Find the orthogonal projection of $\{z_n\}$ onto the vector space spanned by $\{x_n\}$ and $\{y_n\}$.

4
20 points

4. Let $(V, (-,-))$ be a real Euclidean space. Let W be the vector space of linear maps $L : V \to \mathbb{R}$.

4.(a). For each $v \in V$, show that the map of sets

$$(-,v) : V \longrightarrow \mathbb{R}$$
$$(-,v)(v') = (v',v)$$

is a linear map.

4.(b). Show that the map of sets

$$\phi : V \longrightarrow W$$
$$v \mapsto (-,v)$$

is a linear map.

5. True or False.

5
20 points

Give a *brief* explanation for each answer. This *should be at most a sentence or two* with the main idea, or the main theorem you use, or the example, or the counter example, etc.

5.(a). Every complex vector space is isomorphic to $\mathbb{C}^n$ for some nonnegative integer n.

T F

5.(b). Let V be a vector space, and let $V' \subseteq V$ be a subspace. Then the additive identity of V is equal to the additive identity of V'.

T F

5.(c). In the vector space $C^0(\mathbb{R})$ of continuous maps $f : \mathbb{R} \to \mathbb{R}$, the elements $1, \cos t, \sin t$ are linearly independent.

T F

5.(d). A linear map is surjective if and only if it has trivial kernel.

T F

5.(e). Let $(V, (-,-))$ be a Euclidean vector space, and let W be a subspace. Then we have $W \cap W^\perp = \{\mathscr{O}\}$, where $\mathscr{O}$ is the additive identity of V.

T F

5.(f). Let $(V, (-,-))$ be a Euclidean vector space, and let W be a finite dimensional subspace with orthonormal basis $w_1, \ldots, w_n$. Then for any $v \in V$ we have $v - \sum_{i=1}^n (v, w_i) w_i$ is in W.

T F

5.(g). If $v_1, \ldots, v_n$ are nonzero, mutually orthogonal elements of a Euclidean space, then they are linearly indepenent.

T F

5.(h). If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a surjective linear map, then $n \geq m$.

T F

5.(i). A linear map $f : \mathbb{R}^n \to V$ is injective if and only if $\dim \operatorname{Im}(f) = n$.

T F

5.(j). Elements $v_1, \ldots, v_n$ of a vector space V form a basis of V if and only if they span V.

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