Math 2300–007: Exam III Study Guide

Topics List:

- §8.5: Power Series
  - Interval of convergence (Don't forget the endpoints!)
  - Radius of convergence (always symmetric around center)
  - Ratio test
- §8.6: Representations of Functions as Power Series
  - Making new power series from geomentric series by...
    - Substitution
    - Differentiation
    - Integration
    - Multiplication by a power of x
  - Can "steal" the interval of convergence from the geometric series, but *need to check the endpoints!*
- §8.7: Taylor and Maclaurin Series
  - Taylor polynomials/series
    - Find from scratch using f and it's derivatives evaluated at "center" a
    - Find by modifying the Taylor polynomials/series of common functions
  - Remember Maclauren series for $\frac{1}{1-x}$, e^x, $\sin x$, $\cos x$, $\arctan x$, $\ln(1+x)$.
  - Can use Taylor polynomials/series in place of "complicated" functions when e.g.
    - integrating
    - finding limits
    - estimating values such as $\sqrt{2}$ or $\ln(3)$ or e.
  - Taylor's Inequality for estimating error. Use in several ways:
    - to estimate error in estimate of e.g. e or $\sqrt{5}$.
    - to find n so that the error in your approximation is within a certain amount
    - to prove that a Taylor series for $f(x)$ converges to $f(x)$ in the interval of convergence
- §7.1, 7.2, 7.3: Differential Equations (equations involving derivatives)
  - Checking that an equation is a solution to a differential equation
  - Slope fields (graphical perspective)
  - Euler's method (numerical perspective)
  - Separation of variables (analytic perspective)
  - Vocabulary: equilibrium solution (stable, unstable), autonomous

Selected Example Problems Provided by Class

1. Evaluate $\int e^{-x^2}\,dx$ as in infinite series.

2. Use an infinite series to find $\lim\limits_{x\to 0} \dfrac{e^x - 1 - x}{x^2}$.

3. (a) Find the 4th degree polynomial for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.

   (b) Use the previous answer to find the 4th degree Taylor Polynomial for $g(x) = \sqrt{1-x}$ centered at $a = 0$.

   (c) Use the previous answer to find a 3rd degree Taylor Polynomial for $h(x) = \frac{1}{\sqrt{1-x^2}}$

4. Determine the value of the sum $\displaystyle\sum_{n=0}^{\infty} \frac{5\cdot 3^n}{(n+1)!}$.

5. Find the interval of convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n\cdot 3^n}$. What is the radius of convergence?

6. Use Euler's method with a step size of 0.3 to estimate $y(0.9)$, where $y(x)$ is the solution to the initial-value problem
$$y' = x^2,\ y(0) = 1.$$

7. Solve the differential equation
$$f'(x) = f(x)(1 - f(x)),\ f(0) = \frac{1}{3}.$$

8. Find the Taylor series for $f(x) = \frac{7}{x^4}$ about $x = -3$.

9. Show that the Maclauren series for $\cos(x)$ converges to $\cos x$.

10. Estimate $\ln(1.4)$ to be within 0.0005.

11. Solve the differential equation $\frac{dv}{dt} = e^{v+2t}$.

12. Determine the Taylor Series for $f(x) = x^6 e^{2x^3}$ about $x = 0$.

13. A function $y(t)$ satisfies the differential equation
$$\frac{dy}{dt} = y^4 - 6y^3 + 5y^2.$$
What are the constant solutions of the equation? For what values of y is $y(t)$ increasing? Decreasing?

14. Suppose $\sum_{n=0}^{\infty} c_n x^n$ converges when $x = -4$ and diverges when $x = 6$. What can be said about the convergence or divergence of the following series?

    (a) $\sum_{n=0}^{\infty} c_n$

    (b) $\sum_{n=0}^{\infty} c_n 8^n$

    (c) $\sum_{n=0}^{\infty} c_n (-3)^n$

    (d) $\sum_{n=0}^{\infty} (-1)^n c_n 9^n$

15. How large should n be to estimate e to within six decimal places (0.0000005) using a Taylor polynomial centered at $a = 0$?

16. Determine the value of the sum $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

17. What are the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$?

18. Solve the differential equation
$$\frac{dy}{dx} = \frac{te^t}{y\sqrt{1+y^2}}$$

19. Find the 10th degree Taylor Polynomial centered at $a = 1$ for the function $f(x) = 2x^2 - x + 1$.

20. Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}$.

21. Sketch the direction field for the differential equation $y' = xy - x^2$. Then, sketch the solution curve that passes through the point $(0, 1)$.

22. Use a 5th degree Taylor polynomial centered at 0 to estimate $\sin(3°)$. What does Taylor's inequality say about the error?