Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

You see a square root. What now?

$$\int \frac{1}{t\sqrt{t^2-1}}\,dt \qquad \int \frac{1}{\sqrt{4+x^2}}\,dx \qquad \int \frac{1}{x^2\sqrt{1-4x^2}}\,dx$$

We look at Pythagorean trig identities for help.

The goal is to **eliminate the square root** by using a scaled trig identity.

$$1-\cos^2\theta = \sin^2\theta \quad \rightarrow \quad r^2 - r^2\cos^2\theta = r^2\sin^2\theta$$

$$1+\tan^2\theta = \sec^2\theta \quad \rightarrow \quad r^2 + r^2\tan^2\theta = r^2\sec^2\theta$$

$$\sec^2\theta - 1 = \tan^2\theta \quad \rightarrow \quad r^2\sec^2\theta - r^2 = r^2\tan^2\theta$$

5.7 Trigonometric Substitution

$$\int \frac{1}{t\sqrt{t^2-1}}\,dt$$

Inside the square root

t^2-1 looks like $\sec^2\theta - 1$. [This will let us remove the square root by using the identity $\sec^2\theta - 1 = \tan^2\theta$.]

Set $t^2-1 = \sec^2\theta - 1$, which means $t = \sec\theta$. Then $dt = \sec\theta\tan\theta\,d\theta$.

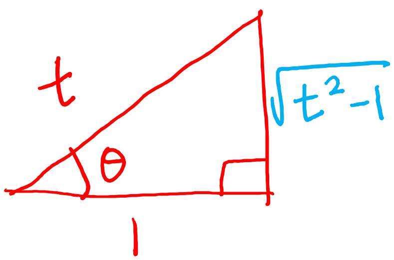
Hence

$$\int \frac{1}{\sec\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta = \int \frac{\tan\theta\,d\theta}{\sqrt{\tan^2\theta}} = \int \frac{\tan\theta}{\tan\theta}\,d\theta = \int d\theta = \theta + C$$

5.7 Trigonometric Substitution

$$\int \frac{1}{t\sqrt{t^2-1}}\,dt$$

How do we convert θ back to t? We create a right triangle using the substitution $t = \sec\theta$. Since $\sec\theta = \frac{t}{1} = \frac{\text{hypotenuse}}{\text{adjacent}}$, we get the triangle



use the pythagorean theorem to compute the length of the unknown side.

Note that we can also use $\theta = \operatorname{arcsec}(t)$, which is easier.

Then $\tan\theta = \frac{\sqrt{t^2-1}}{1}$ so $\arctan\left(\sqrt{t^2-1}\right) = \theta$.

Therefore $\int \frac{1}{t\sqrt{t^2-1}}\,dt = \theta + C = \boxed{\arctan\left(\sqrt{t^2-1}\right) + C \quad (\text{or } \operatorname{arcsec}(t) + C)}$

5.7 Trigonometric Substitution

$$\int \frac{1}{x^2\sqrt{1-4x^2}}\,dx$$

Inside the square root $1-4x^2$ looks like $1-\cos^2\theta$.

set $1-4x^2 = 1-\cos^2\theta$. Then $\cos^2\theta = 4x^2$ so $\cos\theta = 2x$.

$\frac{1}{2}\cos\theta = x$

$-\frac{1}{2}\sin\theta\, d\theta = dx$

$$\int \frac{1}{x^2\sqrt{1-4x^2}}\,dx = \int \frac{1}{\left(\frac{1}{2}\cos\theta\right)^2\sqrt{1-4\left(\frac{1}{2}\cos\theta\right)^2}}\left(-\frac{1}{2}\sin\theta\, d\theta\right)$$

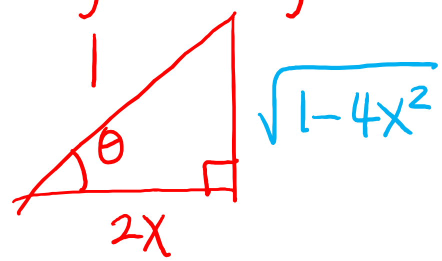
$$= \int \frac{1}{\frac{1}{4}\cos^2\theta\sqrt{1-\cos^2\theta}}\left(-\frac{1}{2}\sin\theta\, d\theta\right) = \int \frac{-2\sin\theta\, d\theta}{\cos^2\theta\sqrt{\sin^2\theta}} = -2\int \sec^2\theta\, d\theta$$

$$= -2\tan\theta + C$$

5.7 Trigonometric Substitution

$$\int \frac{1}{x^2\sqrt{1-4x^2}}\,dx$$

Find $\tan\theta$ in terms of t by creating a right triangle using the substitution equation $\cos\theta = 2x = \frac{2x}{1} = \frac{\text{adjacent}}{\text{hypotenuse}}$



$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{1-4x^2}}{2x}$$

Hence $$\int \frac{1}{x^2\sqrt{1-4x^2}}\,dx = -2\tan\theta + C = -2\,\frac{\sqrt{1-4x^2}}{2x} + C$$

$$= \boxed{\frac{-\sqrt{1-4x^2}}{x} + C}$$

5.7 Trigonometric Substitution

$$\int \frac{1}{\sqrt{4+x^2}}\,dx$$

Inside the square root $4+x^2$ looks like $1+\tan^2\theta$. But we can't set them equal yet. We need to make sure the constant terms are the same. Observe that $4+4\tan^2\theta = 4\sec^2\theta$. Hence $4+x^2 = 4+4\tan^2\theta$ and $x = 2\tan\theta$.

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$$\int \frac{1}{\sqrt{4+4\tan^2\theta}}(2\sec^2\theta\, d\theta) = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$

$$= \int \frac{2\sec^2\theta\, d\theta}{2\sec\theta} = \int \sec\theta\, d\theta$$

5.7 Trigonometric Substitution

$$\int \frac{1}{\sqrt{4+x^2}}\,dx$$

Recall from Week 1 handouts: $\int \sec\theta\, d\theta = \int \frac{\sec\theta\,(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\,d\theta$

$= \int \frac{(\sec^2\theta + \sec\theta\tan\theta)\,d\theta}{\sec\theta+\tan\theta}$

$u = \sec\theta + \tan\theta$

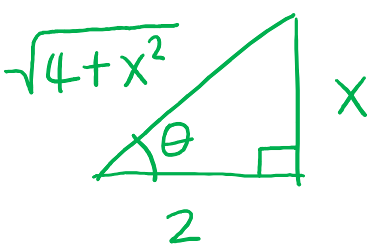
$du = (\sec\theta\tan\theta + \sec^2\theta)\,d\theta$

$$\int \frac{(\sec^2\theta + \sec\theta\tan\theta)\,d\theta}{\sec\theta+\tan\theta} = \int \frac{du}{u} = \ln|u| + C = \ln|\sec\theta + \tan\theta| + C$$

Convert to x

$x = 2\tan\theta$

$\frac{x}{2} = \tan\theta$



$\sec\theta = \frac{\sqrt{4+x^2}}{2}$

$\tan\theta = \frac{x}{2}$

$$= \boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

5.7 Trigonometric Substitution (Completing the Square)

$$\int \frac{dx}{\sqrt{6x - x^2}}$$

Inside the square root, $6x$ is not a constant so we can't (yet) use trig sub. We complete the square to be able to use trig sub.

$$6x - x^2 = -[x^2 - 6x]$$
$$= -[x^2 - 6x + 9 - 9]$$
$$= -[(x-3)^2 - 9]$$
$$= 9 - (x-3)^2$$

Hence we can replace $6x - x^2$ with $9-(x-3)^2$.

$$\int \frac{dx}{\sqrt{6x-x^2}} = \int \frac{dx}{\sqrt{9-(x-3)^2}}$$

Now inside the square root, $9-(x-3)^2$ looks like $9-9\cos^2\theta$. Hence setting them equal, we get

$$9-(x-3)^2 = 9 - 9\cos^2\theta$$
$$(x-3) = 3\cos\theta$$

5.7 Trigonometric Substitution (Completing the Square)

$$\int \frac{dx}{\sqrt{6x - x^2}}$$

$x-3 = 3\cos\theta$

$dx = -3\sin\theta\, d\theta$

$$\int \frac{dx}{\sqrt{9-(x-3)^2}} = \int \frac{-3\sin\theta\, d\theta}{\sqrt{9-9\cos^2\theta}} = \int \frac{-3\sin\theta\, d\theta}{\sqrt{9\sin^2\theta}} = \int \frac{-3\sin\theta\, d\theta}{3\sin\theta} = -\int d\theta$$

$$= -\theta + C$$

solving for θ,

$x-3 = 3\cos\theta$

$\frac{x-3}{3} = \cos\theta$

$\arccos\left(\frac{x-3}{3}\right) = \theta$

Hence

$$\int \frac{dx}{\sqrt{6x-x^2}} = -\theta + C = \boxed{-\arccos\left(\frac{x-3}{3}\right) + C}$$

5.7 Trigonometric Integrals

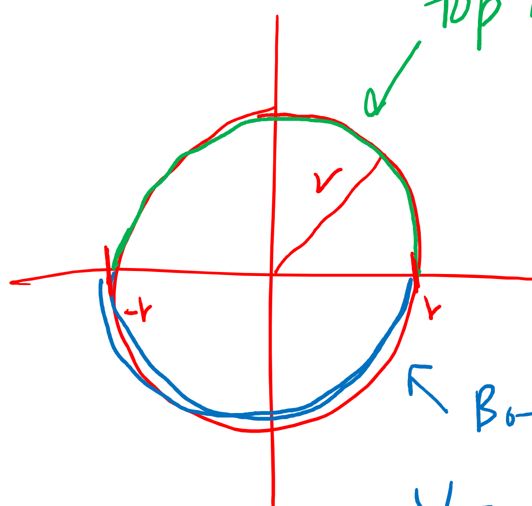
Prove that the area of a circle with radius r is πr^2.

Equation of a circle of radius r centered at the origin

$$x^2+y^2=r^2$$

$$y=\pm\sqrt{r^2-x^2}$$

top half $y=\sqrt{r^2-x^2}$



Bottom half

$$y=-\sqrt{r^2-x^2}$$

$$\text{Area}=\int_a^b(\text{top}-\text{bottom})\,dx$$

$$=\int_{-r}^{r}\sqrt{r^2-x^2}-\left(-\sqrt{r^2-x^2}\right)dx$$

$$=2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

5.7 Trigonometric Integrals

Prove that the area of a circle with radius r is πr^2.

$$2\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$$

$\sin^2\theta = 1-\cos^2\theta$

$r^2\sin^2\theta = r^2 - r^2\cos^2\theta$
$= r^2 - x^2$

$x = r\cos\theta$

$dx = -r\sin\theta\, d\theta$

$$= 2\int_{\theta(-r)=\theta_1}^{\theta(r)=\theta_2}\sqrt{r^2-r^2\cos^2\theta}\,(-r\sin\theta\, d\theta)$$

$$= 2\int_{\theta_1}^{\theta_2}\sqrt{r^2-r^2\cos^2\theta}\,(-r\sin\theta\, d\theta)$$

$$= 2\int_{\theta_1}^{\theta_2}\sqrt{r^2\sin^2\theta}\,(-r\sin\theta\, d\theta)$$

$$= -2\int_{\theta_1}^{\theta_2} r^2\sin^2\theta\, d\theta$$

$$= -2r^2\int_{\theta_1}^{\theta_2}\frac{1-\cos 2\theta}{2}\,d\theta = -r^2\left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

Solving for θ_1 and θ_2: $\frac{x}{r} = \cos\theta$

when $x=r$, $\cos\theta_2 = 1$ so $\theta_2 = 0$.

when $x=-r$, $\cos\theta_1 = -1$ so $\theta_1 = \pi$. Hence

$$\text{Area} = -r^2\left[\left(0 - \cancel{\frac{\sin 0}{2}}^{\,0}\right) - \left(\pi - \cancel{\frac{\sin 2\pi}{2}}^{\,0}\right)\right]$$

$$= -r^2[-\pi] = \pi r^2$$