Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Expanding the Integral

- When we first learned the definite integral, we integrated over a closed interval [a,b].

$$\int_a^b f(x)\ dx$$

- We required that the interval is finite and inside this interval, the function f(x) was continuous and well-defined.
- Question: What if we relax these constraints, as in throw out one or all of these assumptions?

Infinities and Discontinuities

How can we make sense of the integrals below?

$$\int_1^\infty \frac{1}{x}\,dx \qquad \int_1^\infty \frac{1}{x^p}\,dx \qquad \int_{-\infty}^1 \frac{1}{1+x^2}\,dx$$

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx \qquad \int_0^1 \ln x\,dx \qquad \int_0^3 \frac{1}{x-1}\,dx$$

$$\int_{-\infty}^\infty x e^{-x^2}\,dx$$

5.10 Improper Integrals

We call the below integrals "improper" to reflect that we are stretching the original definition of the integral.

Improper Integrals of Type 1

(a)

$$\int_a^\infty f(x)\ dx = \lim_{t\to\infty} \int_a^t f(x)\ dx$$

(b)

$$\int_{-\infty}^b f(x)\ dx = \lim_{t\to-\infty} \int_t^b f(x)\ dx$$

(c)

$$\int_{-\infty}^\infty f(x)\ dx = \int_{-\infty}^a f(x)\ dx + \int_a^\infty f(x)\ dx.$$

In part (c) any real number a can be used. If either of the two limits diverges, then the original integral diverges.

5.10 Improper Integrals

To talk about the finiteness of integrals without doing the actual computation, we use the terms **convergent** to label finite answers and **divergent** to label infinite or undefined answers.

1. Convergent = **finite**

2. Divergent = **infinite** or **undefined**

5.10 Improper Integrals

$$\int_{-\infty}^{1} \frac{1}{1+x^2}\, dx$$

Type 1

$$\int_{-\infty}^{1} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty}\left(\int_t^1 \frac{1}{1+x^2}\,dx\right)$$

$$= \lim_{t\to-\infty}\left(\Big[\arctan(x)\Big]_t^1\right) = \lim_{t\to-\infty}\left(\arctan(1)-\arctan(t)\right)$$

$$= \arctan(1) - \underbrace{\lim_{t\to-\infty}\arctan(t)}_{\text{horizontal asymptote as } t\to-\infty}$$

$$= \frac{\pi}{4} - \left(-\frac{\pi}{2}\right)$$

$$= \frac{3\pi}{4}$$

Therefore

$$\int_{-\infty}^{1}\frac{1}{1+x^2}\,dx = \frac{3\pi}{4}$$

5.10 Improper Integrals

$$\int_{-\infty}^{\infty} xe^{-x^2}\ dx$$

The function $f(x) = xe^{-x^2}$ has no discontinuities.

Type 1

$$\int_{-\infty}^{\infty} xe^{-x^2} dx = \int_{-\infty}^{0} xe^{-x^2} dx + \int_{0}^{\infty} xe^{-x^2} dx$$

$$= \lim_{t \to -\infty} \int_{t}^{0} xe^{-x^2} dx + \lim_{t \to \infty} \int_{0}^{t} xe^{-x^2} dx$$

$$= \lim_{t \to -\infty} \left[\frac{e^{-x^2}}{-2} \right]_t^0 + \lim_{t \to \infty} \left[\frac{e^{-x^2}}{-2} \right]_0^t$$

$$= \lim_{t \to -\infty} \left[\frac{e^0}{-2} + \frac{e^{-t^2}}{2} \right] + \lim_{t \to \infty} \left[\frac{e^{-t^2}}{-2} + \frac{e^0}{2} \right]$$

$$= -\frac{1}{2} + \lim_{t \to -\infty} \frac{1}{2e^{t^2}} + \lim_{t \to \infty} \frac{1}{-2e^{t^2}} + \frac{1}{2}$$

$$= \lim_{t \to -\infty} \frac{1}{2e^{t^2}} + \lim_{t \to \infty} \frac{1}{-2e^{t^2}}$$

$$= \frac{1}{2e^{\infty}} + \frac{1}{-2e^{\infty}}$$

$$= 0 + 0$$

$$= 0.$$

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = 0$$

<u>converges</u>

5.10 Improper Integrals

$$\int_1^\infty \frac{1}{x}\,dx$$

<u>Type 1</u>

$$\int_1^\infty \frac{1}{x}dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x}dx\right) = \lim_{t\to\infty}\left(\Big[\ln|x|\Big]_1^t\right)$$

$$= \lim_{t\to\infty}\left(\ln|t| - \ln|1|\right) = \lim_{t\to\infty}\left(\ln|t| - 0\right) = \lim_{t\to\infty}\ln|t| = \infty$$

Therefore $\int_1^\infty \frac{1}{x}dx$ <u>diverges</u>

5.10 Improper Integrals

$$\int_1^\infty \frac{1}{x^p}\, dx \text{ for } p \neq 1$$ Type 1

$$\int_1^\infty \frac{1}{x^p}dx = \lim_{t\to\infty}\left(\int_1^t \frac{1}{x^p}dx\right) = \lim_{t\to\infty}\left(\left[\frac{x^{1-p}}{1-p}\right]_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{t^{1-p}}{1-p} - \frac{1^{1-p}}{1-p}\right) = \lim_{t\to\infty}\left(\frac{1}{p-1} - \frac{t^{1-p}}{p-1}\right)$$

$$= \frac{1}{p-1} - \frac{1}{p-1}\left(\lim_{t\to\infty} t^{1-p}\right)$$

$$= \begin{cases} \frac{1}{p-1} - 0 & \text{(convergent) if } p>1 \\ \frac{1}{p-1} + \infty & \text{(divergent) if } p<1 \end{cases}$$

p-test

$\int_1^\infty \frac{1}{x^p}dx$ converges for $p>1$

diverges for $p \leq 1$

5.10 P-test

$\boxed{2}$ $$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

- This is super important! This idea will used over and over for the rest of the semester.

5.10 Improper Integrals

Comparing $\frac{1}{x^2}$ and $\frac{1}{x}$: It can be hard to see whether an integral converges or diverges from just looking at the graph.

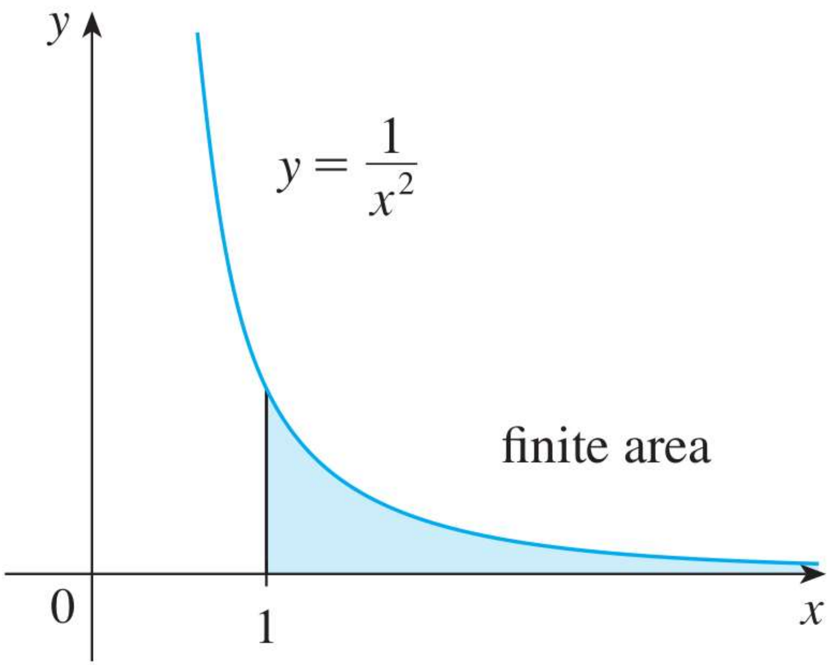


FIGURE 4 $\int_1^\infty (1/x^2)\,dx$ converges

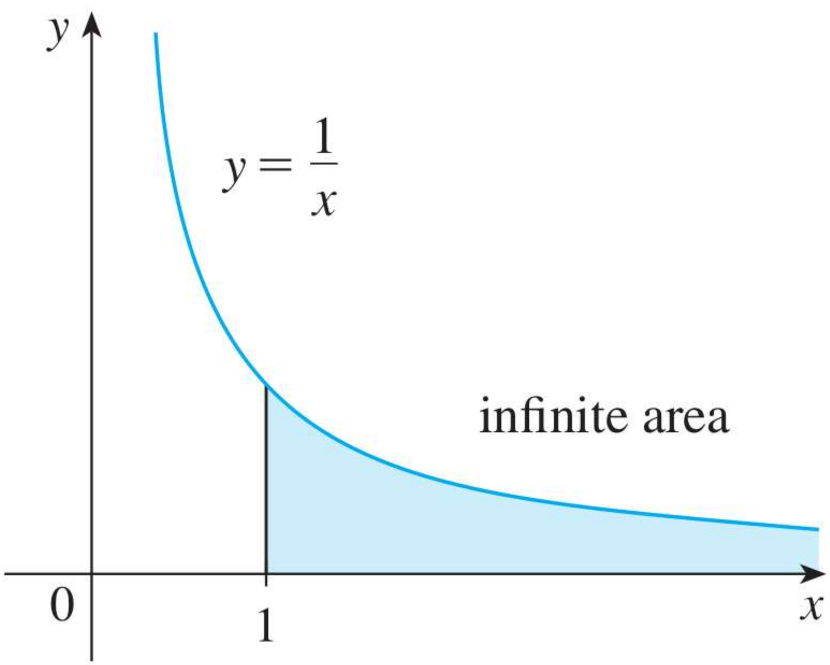
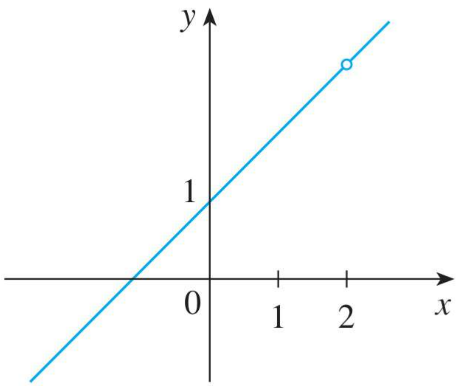


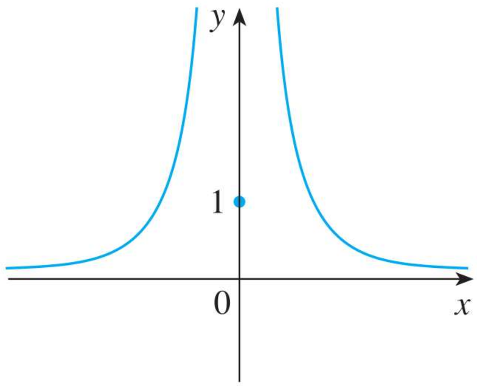
FIGURE 5 $\int_1^\infty (1/x)\,dx$ diverges

Types of Discontinuities



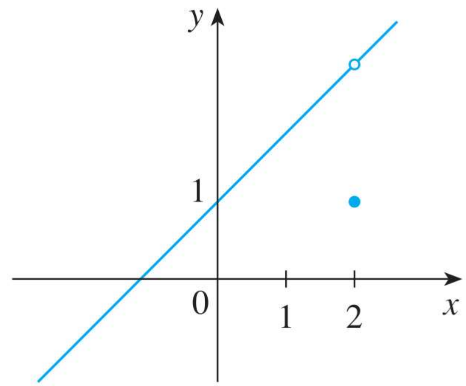
(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

Removable Discontinuity



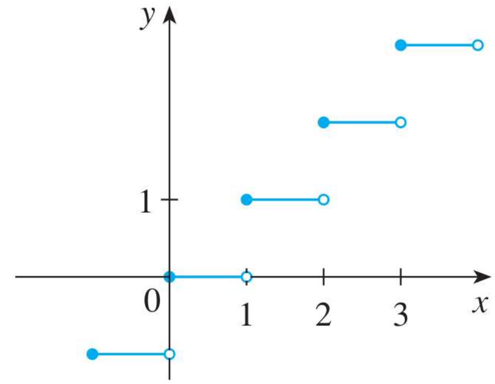
(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Infinite Discontinuities



(c) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$

Removable Discontinuity



(d) $f(x) = [\![x]\!]$

Jump Discontinuities

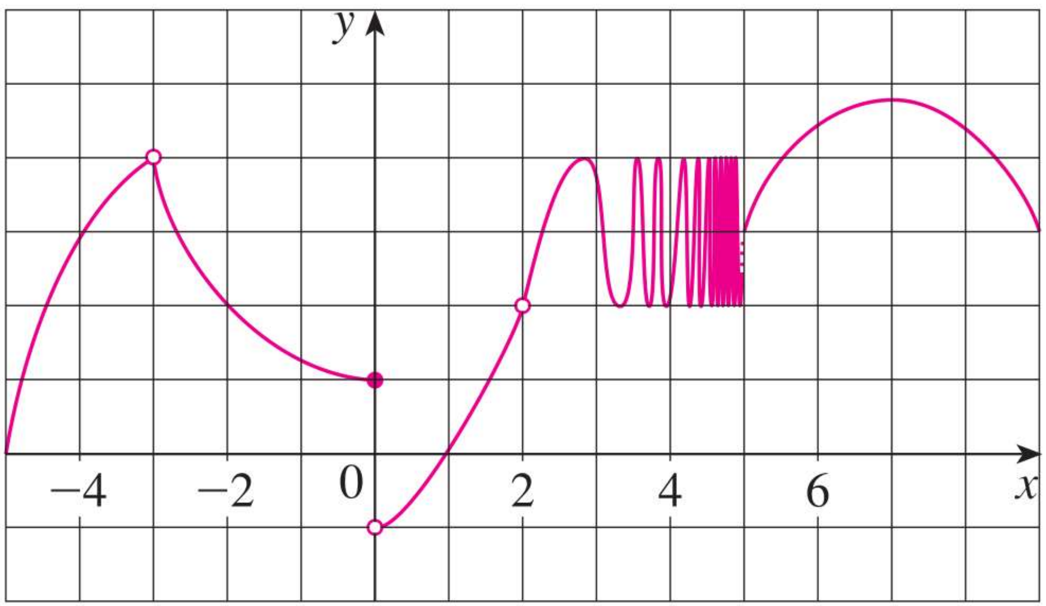
Identifying Discontinuities from a Graph

(a) $\lim_{x \to -3^-} h(x)$ (b) $\lim_{x \to -3^+} h(x)$ (c) $\lim_{x \to -3} h(x)$

(d) $h(-3)$ (e) $\lim_{x \to 0^-} h(x)$ (f) $\lim_{x \to 0^+} h(x)$

(g) $\lim_{x \to 0} h(x)$ (h) $h(0)$ (i) $\lim_{x \to 2} h(x)$

(j) $h(2)$ (k) $\lim_{x \to 5^+} h(x)$ (l) $\lim_{x \to 5^-} h(x)$



Identifying Discontinuities from a Formula

1. Identify the domain of the function.
2. Look at the function and see if there are any values of x for which we **divide by zero** or when the function is **undefined**.
3. If any of the x-values from step 2 are inside the domain, then the function has discontinuities at those values of x.

Example: $f(x) = \dfrac{1}{x(x+1)} \qquad D : [0, \infty)$

Example: $f(x) = \ln(x) \qquad D : [0, 3]$

5.10 Improper Integrals

Improper Integrals of Type 2.

- First look at the interval that you are integrating over.
- Determine if your function has any discontinuities (For example, asymptotes, which are characterized by dividing by zero or undefined)

(a) If f is continuous on $[a, b)$ and **discontinuous** at $x = b$:

$$\int_a^b f(x)\, dx = \lim_{t \to b^-} \int_a^t f(x)\, dx$$

(b) If the **discontinuity** is at the lower endpoint $x = a$:

$$\int_a^b f(x)\, dx = \lim_{t \to a^+} \int_t^b f(x)\, dx$$

(c) If the integral has a **discontinuity** in the middle of the interval or has **discontinuities** at both ends, break the integral into two.

(c) Discontinuity in the middle

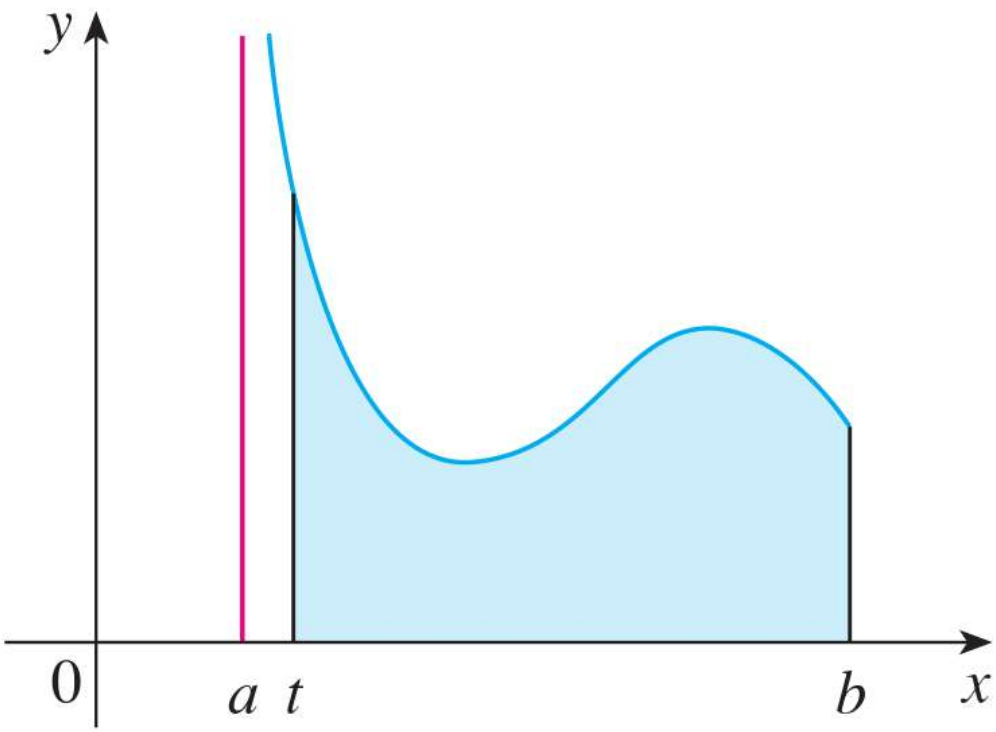
- Split your integral into two at the discontinuity as shown below:

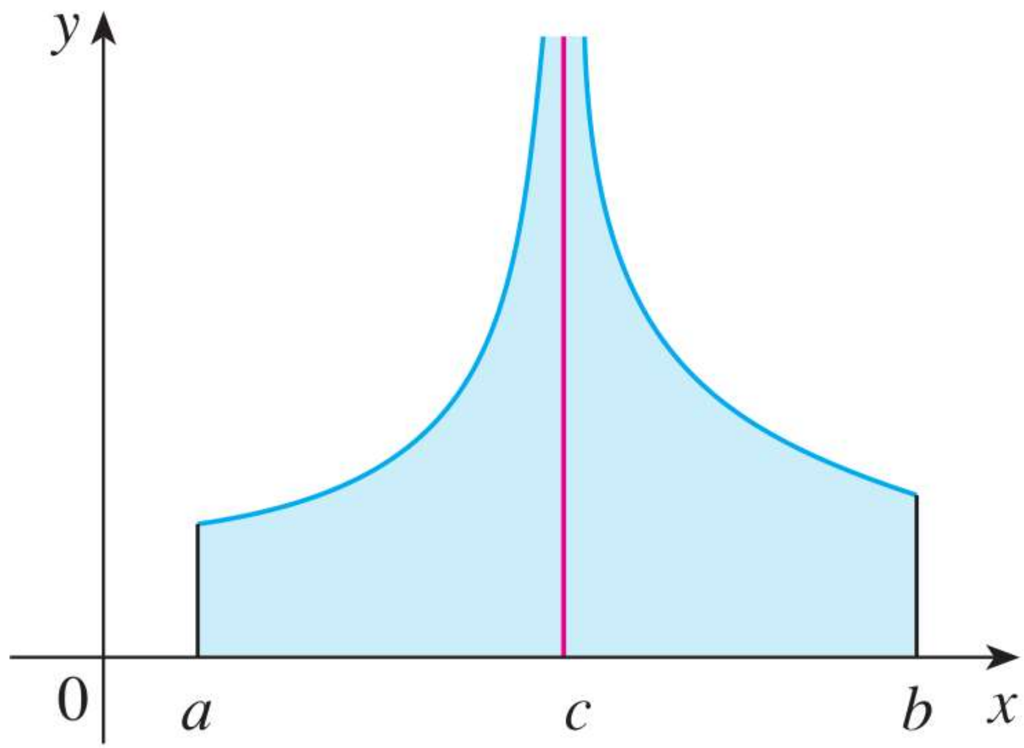
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

$$= \lim_{t \to c^-} \int_a^t f(x)\,dx + \lim_{t \to c^+} \int_t^b f(x)\,dx$$

If either of the two limits diverges, then the original integral diverges.

5.10 Improper Integrals





From now on, whenever you see a **definite integral**, check for any **discontinuities inside the interval** you are integrating over. If there are any, it will be an improper integral.

Graphs of the examples

5.10 Improper Integrals

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,dx$$

<u>Type 2</u>

Discontinuity at $x=1$, which is on the boundary of $[1,2]$.

substitution with $u=x-1$

$$\int_1^2 \frac{1}{\sqrt{x-1}}dx = \lim_{t\to 1^+}\left(\int_t^2 \frac{dx}{\sqrt{x-1}}\right) = \lim_{t\to 1^+}\left(\left[2(x-1)^{\frac{1}{2}}\right]_t^2\right)$$

$$= \lim_{t\to 1^+}\left(2(2-1)^{\frac{1}{2}} - 2(t-1)^{\frac{1}{2}}\right) = 2 - 2\lim_{t\to 1^+}\sqrt{t-1} = 2-0 = 2$$

Therefore $\displaystyle\int_1^2 \frac{1}{\sqrt{x-1}}dx = 2$

5.10 Improper Integrals

$$\int_0^1 \ln x \; dx$$

Type 2

Discontinuity at $x=0$ on the boundary of $[0,1]$.

$$\int_0^1 \ln x\,dx = \lim_{t\to 0^+}\left(\int_t^1 \ln x\,dx\right) = \lim_{t\to 0^+}\left(\Big[x\ln x - x\Big]_t^1\right)$$

$$= \lim_{t\to 0^+}\left(\left(\cancel{\ln 1}^{\,0} - 1\right) - \left(t\ln t - t\right)\right) = -1 - \lim_{t\to 0^+}\left(t\ln t - t\right)$$

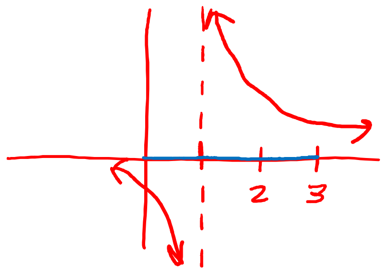
$$\lim_{t\to 0^+} t\ln t = \lim_{t\to 0^+}\frac{\ln t}{\frac{1}{t}} \overset{L'H}{=} \lim_{t\to 0^+}\frac{\frac{1}{t}}{-\frac{1}{t^2}} = \lim_{t\to 0^+} -t = 0$$

$$\lim_{t\to 0^+}(t\ln t - t) = 0 - 0 = 0.$$

Therefore $\displaystyle\int_0^1 \ln x\,dx = -1 - 0 = -1.$

5.10 Improper Integrals

$$\int_0^3 \frac{1}{x-1}\,dx$$



The function $f(x)=\frac{1}{x-1}$ has a discontinuity (vertical asymptote) at $x=1$, which is inside the interval $[0,3]$ that we are integrating over.

Improper integral of Type 2

Split the integral at the discontinuity.

$$\int_0^3 \frac{1}{x-1}dx = \underbrace{\int_0^1 \frac{1}{x-1}dx}_{\text{Type 2}} + \underbrace{\int_1^3 \frac{1}{x-1}dx}_{\text{Type 2}} = \lim_{t\to 1^-}\int_0^t \frac{1}{x-1}dx + \lim_{t\to 1^+}\int_t^3 \frac{1}{x-1}dx$$

Type 2
Discontinuity on the right endpoint

Type 2
Discontinuity on the left endpoint

approach 1 from the left

approach 1 from the right

Compute the limits

$$\lim_{t\to 1^-}\int_0^t \frac{1}{x-1}\,dx = \lim_{t\to 1^-}\left(\Big[\ln|x-1|\Big]_0^t\right) = \lim_{t\to 1^-}\left(\ln|t-1| - \ln|0-1|\right) = \lim_{t\to 1^-}\left(\ln|t-1| - \cancel{\ln 1}^{\,0}\right)$$

$= -\infty$ Since one of the two integrals diverges, we can stop and say that our original integral $\int_0^3 \frac{1}{x-1}\,dx$ diverges.

Just to show the other computation:

$$\lim_{t\to 1^+}\int_t^3 \frac{1}{x-1}\,dx = \lim_{t\to 1^+}\left(\ln|3-1| - \ln|t-1|\right) = \ln(2) - (-\infty) = \infty$$

The other integral also diverges.

Warning

$$\int_0^3 \frac{1}{x-1}\,dx = -\infty + \infty \neq 0$$ because we can't add infinities like numbers.

5.10 Improper Integrals (Asymptotes)

Warning: If we had not noticed the asymptote $x = 1$ in Example 7 and had instead confused the integral with an ordinary integral, then we might have made the following erroneous calculation:

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\Big]_0^3 = \ln 2 - \ln 1 = \ln 2$$

This is wrong because the integral is improper and must be calculated in terms of limits.