Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Reminders

- Take-home Quiz 2 is due in two weeks, October 19th.
- The second exam is on October 22nd.

8.2 Geometric Series

Consider the **geometric** <u>**sequence**</u> $\{ar^n\}$. The series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is called the **geometric** <u>**series**</u> with common ratio r and initial value a.

$$S_k = \sum_{n=0}^{k} ar^n = a + ar + ar^2 + \cdots + ar^k$$

$$S_{k+1} = \sum_{n=0}^{k+1} ar^n = a + ar + ar^2 + \cdots + ar^k + ar^{k+1}$$

$$S_{k+1} = a + rS_k$$

$$S_{k+1} - S_k = a + rS_k - S_k$$

$$ar^{k+1} = a + S_k(r-1)$$

$$\frac{ar^{k+1} - a}{r-1} = S_k$$

$$\frac{a(r^{k+1} - 1)}{r-1} = S_k$$

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{a(r^{k+1}-1)}{r-1}$$

If $|r|<1$, $\lim_{k\to\infty} r^{k+1} = 0$

so $\sum_{n=0}^{\infty} ar^n = \frac{a(0-1)}{r-1} = \frac{a}{1-r}$

if $|r|<1$. For $|r| \geq 1$, the geometric series diverges.

8.2 Geometric Series

The geometric series with initial value a and common ratio r

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots$$

is **convergent** if $|r| < 1$ and its sum is

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is **divergent.**

- In a geometric series, the initial value a is **always** the first term.
- To find the common ratio r, take the ratio of two consecutive terms

$$\frac{a_{n+1}}{a_n} = \frac{ar^{n+1}}{ar^n} = r$$

8.2 Geometric Series

Determine whether the series below converges or diverges. If it is convergent, find its sum.

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

Intuition:

The sum in decimal is $0.9 + 0.09 + 0.009 + \cdots$

$= 0.99\overline{9}$ (repeating)

Is this number different from 1?

Let's take their difference

$1 - 0.9 = 0.1$

$1 - 0.99 = 0.01$

$1 - 0.999 = 0.001$

$\vdots$

$1 - 0.99\overline{9} = 0.00\overline{0}$

Here the digit 1 is being pushed further and further into infinity.

Since $0.00\overline{0} = 0$, $1 = 0.99\overline{9}$.

Now let's actually do the work.

First find the summation formula for the series.

① Numerator

$9, 9, 9, \cdots$

constant so 9 matches the pattern.

② Denominator

$10, 100, 1000, \cdots$

If we start at $n=1$, 10^n matches the pattern.

8.2 Geometric Series

Determine whether the series below converges or diverges. If it is convergent, find its sum.

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$$

Since the series isn't alternating (no negative signs), we can skip introducing $(-1)^n$.

So the formula for the terms is

$$a_n = \frac{9}{10^n}$$

and the series is

$$\sum_{n=1}^{\infty} \frac{9}{10^n}.$$

Observe that the series appears to be a geometric series so let's find a and r.

a is the first term so $a = \frac{9}{10}$.

$$r = \frac{a_{n+1}}{a_n} = \frac{9/10^{n+1}}{9/10^n} = \frac{1}{10}.$$

Since $|r| = \left|\frac{1}{10}\right| < 1$, the geometric series converges and

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{a}{1-r} = \frac{9/10}{1-\frac{1}{10}} = \frac{9/10}{9/10} = 1.$$

So the series converges to 1.

8.2 Geometric Series

Determine whether the series below converges or diverges. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Observe that $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ so this series is geometric.

The first term of the series is $a = \left(\frac{1}{2}\right)^1 = \frac{1}{2}$ and $r = \frac{\left(\frac{1}{2}\right)^{n+1}}{\left(\frac{1}{2}\right)^n} = \frac{1}{2}$

and since $-1 < r < 1$, the geometric series converges to

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1.$$

8.2 Geometric Series

EXAMPLE 4 **Expressing a repeating decimal as a rational number**

Write the number $2.3\overline{17} = 2.3171717...$ as a ratio of integers.

First observe that $2.31717\overline{17} = 2.3 + 0.017 + 0.00017 + 0.0000017 + \cdots$

$$= 2.3 + \frac{17}{1000} + \frac{17}{100000} + \frac{17}{10000000} + \cdots$$

2.3 is not repeated so let's write only the repeating decimals as a series.

$$\frac{17}{1000} + \frac{17}{100000} + \frac{17}{10000000} + \cdots$$

To find the formula for a_n,

① Numerator

$17, 17, \cdots$

constant so 17 matches the pattern.

② Denominator

$1000, 100000, 10000000, \cdots$

$10^3, 10^5, 10^7, \cdots$

odd powers start at 3 so 10^{2n+3} matches the pattern if we start at $n=0$.

8.2 Geometric Series

EXAMPLE 4 Expressing a repeating decimal as a rational number

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

3 The series is not alternating so no need for $(-1)^n$.

4 Combine

$$a_n = \frac{17}{10^{2n+3}} \quad \text{where } n = 0, 1, 2, \ldots$$

Then the series for the repeating decimals is

$$0.0\overline{17} = \sum_{n=0}^{\infty} \frac{17}{10^{2n+3}}.$$

This series appears to be a geometric series. Let's find a and r.

a is the first term so $a = 0.017 = \frac{17}{1000}$.

Then

$$r = \frac{a_{n+1}}{a_n} = \frac{17/10^{2(n+1)+3}}{17/10^{2n+3}} = \frac{17/10^{2n+5}}{17/10^{2n+3}}$$

$$= \frac{10^{2n+3}}{10^{2n+5}} = \frac{1}{100}.$$

Since $|r| = \left|\frac{1}{100}\right| < 1$, the series converges and

$$\sum_{n=0}^{\infty} \frac{17}{10^{2n+3}} = \frac{a}{1-r} = \frac{17/1000}{1-\frac{1}{100}} = \frac{17}{990}$$

8.2 Geometric Series

EXAMPLE 4 Expressing a repeating decimal as a rational number

Write the number $2.3\overline{17} = 2.3171717\ldots$ as a ratio of integers.

Therefore $2.3\overline{17} = 2.3 + \sum_{n=0}^{\infty} \frac{17}{10^{2n+3}}$

$$= \frac{23}{10} + \frac{17}{990}$$

$$= \frac{23(99)+17}{990}$$

$$= \frac{2294}{990}.$$

8.2 Geometric Series

Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

First let's find the formula for the terms of the series.

1. Numerator

$5, 10, 20, 40, \cdots$

$5 \cdot 2^n$ matches the pattern if we start at $n=0$.

2. Denominator

$1, 3, 9, 27, \cdots$

3^n matches the pattern if we start at $n=0$.

3. Alternating signs

$1, -1, 1, -1, \cdots$

$(-1)^n$ matches the pattern if we start at $n=0$.

4. Combine

$$a_n = \frac{5 \cdot 2^n}{3^n} \cdot (-1)^n \quad \text{for } n = 0, 1, 2, \cdots$$

So the series is $\displaystyle\sum_{n=0}^{\infty} \frac{5 \cdot (-1)^n \cdot 2^n}{3^n}$

8.2 Geometric Series

Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$$

If we combine the terms with the power n, we get

$$\sum_{n=0}^{\infty} \frac{5 \cdot (-1)^{n} 2^{n}}{3^{n}} = \sum_{n=0}^{\infty} 5 \cdot \left(\frac{-2}{3}\right)^{n}$$

This appears to be a geometric series.
Let's find a and r.

a is the first term so $a = 5$.

$$r = \frac{a_{n+1}}{a_n} = \frac{5\left(-2/3\right)^{n+1}}{5\left(-2/3\right)^{n}} = -\frac{2}{3}$$

Since $|r| = |-2/3| < 1$, the geometric series converges and

$$\sum_{n=0}^{\infty} 5\left(-2/3\right)^{n} = \frac{a}{1-r} = \frac{5}{1-\left(-\frac{2}{3}\right)}$$

$$= \frac{5}{5/3} = 3.$$

Hence the series converges to 3.

8.2 Geometric Series

Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

First, let's combine the terms with power n.

$$a_n = 2^{2n}\,3^{1-n} = (2^2)^n \cdot 3^1 \cdot 3^{-n}$$
$$= 4^n \cdot 3^1 \cdot (3^{-1})^n$$
$$= 3 \cdot \left(\frac{4}{3}\right)^n.$$

The series appears to be geometric.

Let's find a and r.

a is the first term and since the counting variable n starts at 1, $a = a_1 = 3 \cdot \left(\frac{4}{3}\right)^1$
$= 4.$

Computing r,

$$r = \frac{a_{n+1}}{a_n} = \frac{3\,(4/3)^{n+1}}{3\,(4/3)^n} = 4/3$$

Since $|r| = \left|\frac{4}{3}\right| \geq 1$, the geometric series diverges.

8.2 Geometric Series

For what values of x does the series $\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n}$ converge?

Recall: The geometric series $\sum_{n=0}^{\infty} ar^n$ converges to $\frac{a}{1-r}$ when $|r|<1$.

Note:

$$\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n} = \sum_{n=0}^{\infty} \left(\frac{x+5}{3}\right)^n.$$

Compute a and r.

a is the first term. Since n starts at 0,

$$a = \left(\frac{x+5}{3}\right)^0 = 1 \quad \text{and}$$

$$r = \frac{a_{n+1}}{a_n} = \frac{\left(\frac{x+5}{3}\right)^{n+1}}{\left(\frac{x+5}{3}\right)^n} = \frac{x+5}{3}.$$

Since the geometric series converges when $|r|<1$, we have that $\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n}$ converges when $\left|\frac{x+5}{3}\right| < 1$. Solving the inequality,

$$\left|\frac{x+5}{3}\right| = \frac{|x+5|}{3} < 1$$

$$|x+5| < 3$$

$|x-(-5)| < 3$. This is an interval centered at -5 with radius 3.

For what values of x does the series $\displaystyle\sum_{n=0}^{\infty} \frac{(x+5)^n}{3^n}$ converge?

So for $x \in (-8, -2)$, the series converges.

8.2 Sums of Series

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

(i) $\displaystyle\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\displaystyle\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\displaystyle\sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

8.2 Sums of Series

$a + ar + ar^2 + ar^3 + \cdots$

Find the sum of the series $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$. $\overset{?}{=} \underbrace{\sum_{n=1}^{\infty}\frac{3}{n(n+1)}}_{1} + \underbrace{\sum_{n=1}^{\infty}\frac{1}{2^n}}_{2}$

1 Since $\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 1$, $\sum_{n=1}^{\infty}\frac{3}{n(n+1)} = 3\sum_{n=1}^{\infty}\frac{1}{n(n+1)} = 3\cdot 1 = 3$.

2 Geometric series

with $a = \frac{1}{2}$, $r = \frac{1}{2}$

$$\sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{a}{1-r} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1$$

So the limit laws apply and

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right) = \sum_{n=1}^{\infty}\frac{3}{n(n+1)} + \sum_{n=1}^{\infty}\frac{1}{2^n}$$

$$= 3 + 1 = 4.$$