Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Let $f(x) = 1 + x + x^2$. Find the 2nd degree Taylor polynomial of $f(x)$ centered at 1.

Approximating functions using polynomials is very useful!

Example: Approximating $f(x) = \cos x$ with Taylor polynomials centered at 0.

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots + \frac{f^{(k)}(0)}{k!} x^k$$

https://www.desmos.com/calculator/chybqs87ex

Observation: We get better and better approximation as we increase the degree k of the Taylor polynomial $T_k(x)$.

What happens if we let $k \to \infty$? Can a function be **equal** to the limit of its Taylor polynomials?

In some cases, a function is indeed equal to its **Taylor series** $T(x)$. We define the **Taylor series of a function** $f(x)$ **centered at 0** as

$$T(x) = \lim_{k\to\infty} T_k(x)$$

$$= \lim_{k\to\infty} \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!} x^n$$

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

To study Taylor series, we need to first understand the properties of a more general mathematical object called the **Power Series**.

8.5 Power Series

A **power series** (centered at 0) is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

where x is a variable and the c_n's are constants called the **coefficients** of the series.

A power series may converge for some values of x and diverge for other values of x. Note that **the power series resembles a polynomial.** The only difference is that it has infinitely many terms.

8.5 Power Series

Example. The Taylor series centered at 0,

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

is a type of a power series where $c_n = \dfrac{f^{(n)}(0)}{n!}$.

Example. If we take $c_n = 1$ for all n, the power series becomes the **geometric series**

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$$

For what values of x is the geometric series convergent?

Ratio Test for Power Series Centered at 0.

Given a power series $\sum_{n=0}^{\infty} c_n x^n$, let $L(x) = \lim_{n\to\infty} \left| \frac{c_{n+1}x^{n+1}}{c_n x^n} \right| = |x| \cdot \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$.

1. For values of x where $L(x) < 1$, the power series is absolutely convergent.
2. For values of x where $L(x) > 1$, the power series diverges.
3. For values of x where $L(x) = 1$, the Ratio Test is inconclusive and we must use other testing methods to determine convergence.

8.5 Power Series

Theorem. For a given power series (centered at 0) $\sum_{n=0}^{\infty} c_n x^n$ there are only three possibilities:

1. The series converges only when $x = 0$. ($R = 0$)
2. The series converges for all x. ($R = \infty$)
3. There is a positive number R such that the series converges if $|x| < R$ and diverges if $|x| > R$.

Definition. The number R is called the **radius of convergence** of the power series.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

8.5 Power Series – Finding the Radius of Convergence

The Ratio Test is used to determine the radius of convergence R.

The Ratio Test is inconclusive on the boundary of the interval of convergence, so the **endpoints must be checked with other tests** such as the divergence test, integral test, direct comparison test, limit comparison test, or the alternating series test.

8.5 Power Series

V EXAMPLE 1 A power series that converges only at its center

For what values of x is the series $\sum_{n=0}^{\infty} n!x^n$ convergent?

8.5 Power Series

EXAMPLE 3 **A power series that converges for all values of x** Find the domain of the Bessel function of order 0 defined by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$$

8.5 Power Series

Definition. A series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

is called a **power series centered at** a.

Ratio Test for Power Series Centered at a.

Given a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, let

$$L(x) = \lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\cdot\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|.$$

1. For values of x where $L(x) < 1$, the power series is absolutely convergent.
2. For values of x where $L(x) > 1$, the power series diverges.
3. For values of x where $L(x) = 1$, the Ratio Test is inconclusive and we must use other testing methods to determine convergence.

8.5 Power Series

Theorem. For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only three possibilities:

1. The series converges only when $x = a$. $(R = 0)$
2. The series converges for all x. $(R = \infty)$
3. There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

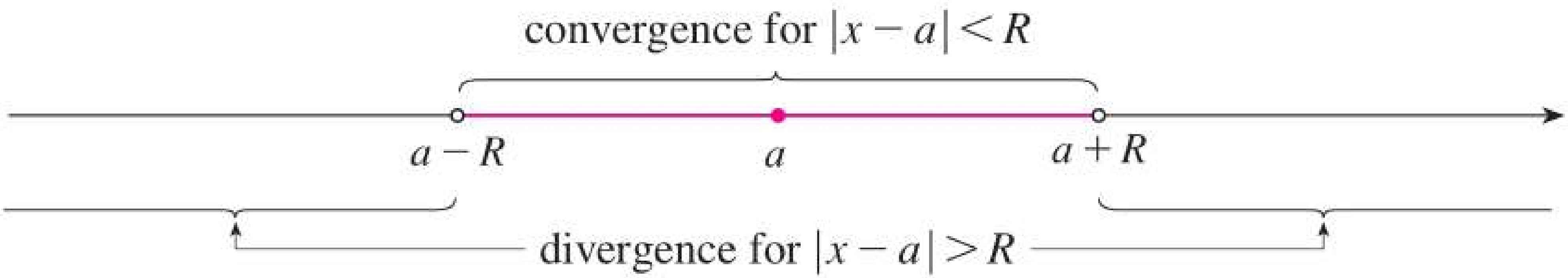
Definition. The number R is called the **radius of convergence** of the power series.

The **interval of convergence** of a power series is the interval that consists of all values of x for which the series converges.

8.5 Power Series

WARNING: When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen - the series might converge at one or both endpoints or it might diverge at both endpoints. Thus when R is positive and finite, there are four possibilities for the interval of convergence:

$$(a - R, a + R), \quad (a - R, a + R], \quad [a - R, a + R), \quad [a - R, a + R]$$



8.5 Power Series

EXAMPLE 2 **Using the Ratio Test to determine where a power series converges**

For what values of x does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

8.5 Power Series

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8.5 Power Series

	Series	Radius of convergence	Interval of convergence
Geometric series	$\sum_{n=0}^{\infty} x^n$	$R = 1$	$(-1, 1)$
Example 1	$\sum_{n=0}^{\infty} n!\, x^n$	$R = 0$	$\{0\}$
Example 2	$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$	$R = 1$	$[2, 4)$
Example 3	$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$	$R = \infty$	$(-\infty, \infty)$