Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

8.3 Direct Comparison Test

Direct Comparison Test.

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $0 \leq a_n \leq b_n$ for all n. Then

$$0 \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$$

and

(a) If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

To use either the Direct Comparison Test or the Limit Comparison Test, we need to compare our messy-looking series to another series that we already understand. Below are the series that we understand so far:

1. A geometric series (a and r are constants)

$$\sum_{n=0}^{\infty} ar^n$$

2. A p-series (p is a constant)

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

3. A series that looks similar to an improper integral that can be solved using u-sub or other integration techniques

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} \approx \int_2^{\infty} \frac{1}{x \ln x}\, dx$$

8.3 Direct Comparison Test

Test the series $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ for convergence or divergence.

We know from using the Limit Comparison Test that the series $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ converges. But can we use the Direct Comparison Test instead? Observe that $\frac{1}{2^n-1} \leq \frac{1}{2^{n-1}}$ for $n \geq 1$ (verify this by graphing or by multiplying both sides of the inequality by 2^{n-1}).

Then $\sum_{n=1}^{\infty} \frac{1}{2^n-1} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ and since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges (geometric, $r = \frac{1}{2}$), $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$ also converges.

Let's show the work. First check the hypothesis for the Direct Comparison Test.

Hypothesis: Both $\frac{1}{2^n-1}$ and $\frac{1}{2^{n-1}}$ are positive for $n \geq 1$.

Now we proceed with DCT.

Since $\frac{1}{2^n-1} \leq \frac{1}{2^{n-1}}$ for $n \geq 1$, $\sum_{n=1}^{\infty} \frac{1}{2^n-1} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$

and since the series on the right hand side $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ converges

(geometric, $r=\frac{1}{2}$), by the Direct Comparison Test $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

converges.

8.3 Direct Comparison Test

Determine whether the series $\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$ converges or diverges.

Observe that $\frac{5}{2n^2+4n+3} \le \frac{5}{2n^2}$ for $n \ge 1$.

Since both of the terms above are positive for $n \ge 1$, we can use the DCT. Then

$$0 \le \frac{5}{2n^2+4n+3} \le \frac{5}{2n^2} \quad (\text{for } n \ge 1)$$

implies

$$0 \le \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \le \sum_{n=1}^{\infty} \frac{5}{2n^2}.$$

If the series on the right converges, we can conclude that the middle series also converges.

Note that $\sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a p-series with $p = 2$. Since $p = 2 > 1$, by p-test $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and so $\frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Hence by the DCT,

$$\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \text{ converges.}$$