# Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

# 8.1 Sequences

A **sequence** is a list of numbers written in a definite order:

$$a_1, a_2, a_3, a_4, \cdots, a_n, \cdots$$

The elements $a_n$ are called the **terms** of the sequence. Note that sequences don't end, and the terms $a_1, a_2, a_3, \cdots$ need not be distinct.

Given a sequence, it is customary to use $\{a_n\}$ instead of $a_1, a_2, a_3, \cdots$.

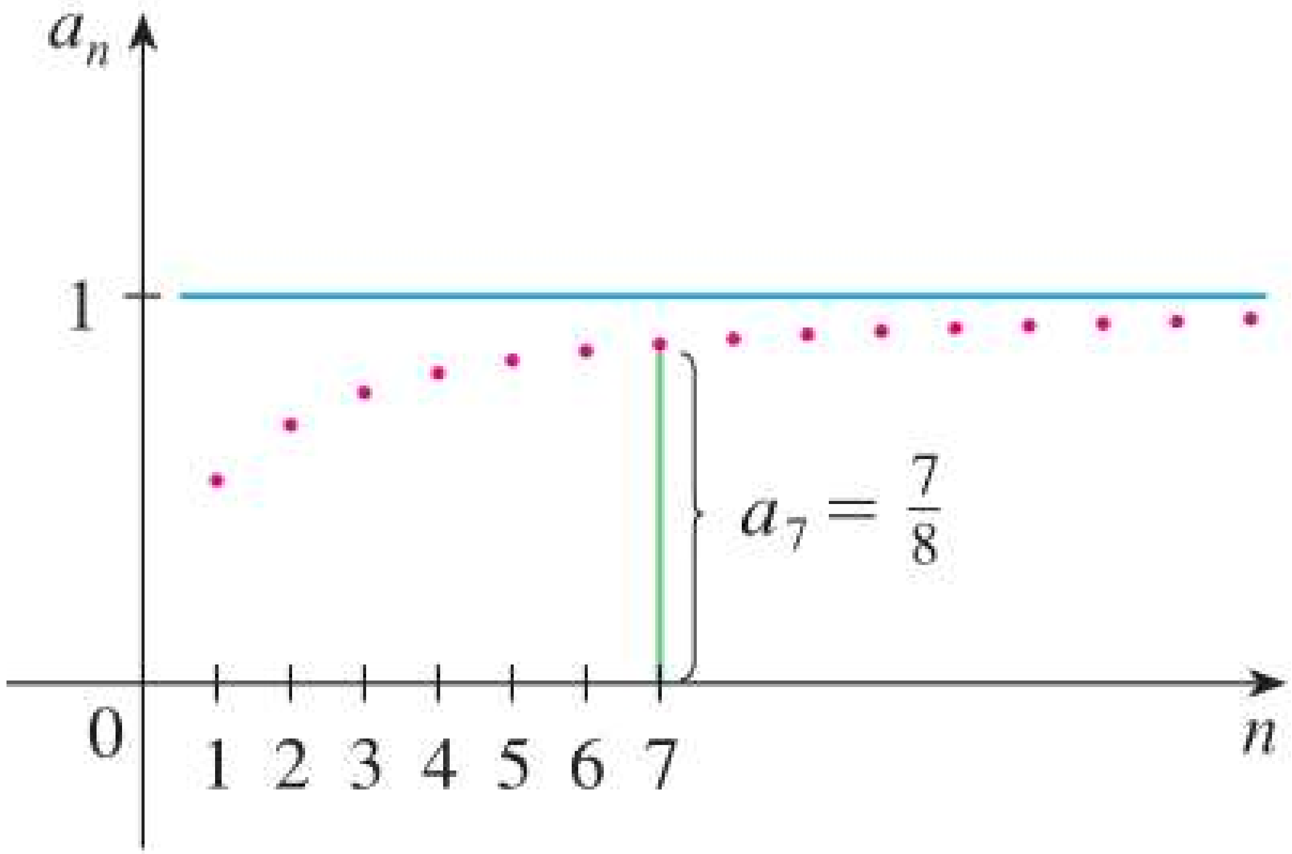
# Examples:

| Standard form | Expanded form | Formula |
|---|---|---|
| $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$ | $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots, \frac{n}{n+1}, \dots\right\}$ | $a_n = \frac{n}{n+1}$ |
| $\left\{\frac{(-1)^n(n+1)}{3^n}\right\}_{n=1}^{\infty}$ | $\left\{-\frac{2}{3}, \frac{3}{9}, -\frac{4}{27}, \frac{5}{81}, \dots, \frac{(-1)^n(n+1)}{3^n}, \dots\right\}$ | $a_n = \frac{(-1)^n(n+1)}{3^n}$ |
| $\left\{\sqrt{n-3}\right\}_{n=3}^{\infty}$ | $\{0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots\}$ | $a_n = \sqrt{n-3},\ n \geqslant 3$ |
| $\left\{\cos\frac{n\pi}{6}\right\}_{n=0}^{\infty}$ | $\left\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \dots, \cos\frac{n\pi}{6}, \dots\right\}$ | $a_n = \cos\frac{n\pi}{6},\ n \geqslant 0$ |

# 8.1 Sequences

- What does a sequence look like?

$$a_n = \frac{n}{n+1}$$



# 8.1 Sequences

Find a formula for the general term $a_n$ of the sequence

$$\left\{ \underbrace{\frac{3}{5}}_{}^{n=1}, -\frac{4}{25}^{\,n=2}, \frac{5}{125}^{\,n=3}, -\frac{6}{625}^{\,n=4}, \frac{7}{3125}, \dots \right\}$$

① numerator

3, 4, 5, 6, 7, . . .

linear behavior so equation of a line.

$n+2$ matches the pattern.

② Denominator.

5, 25, 125, 625, . . .

$5^1, 5^2, 5^3, 5^4, \dots$

exponential behavior

$5^n$

③ negative signs

+, −, +, −, +, −, . . .

alternating behavior

$(-1)^{n+1}$

④ Combine

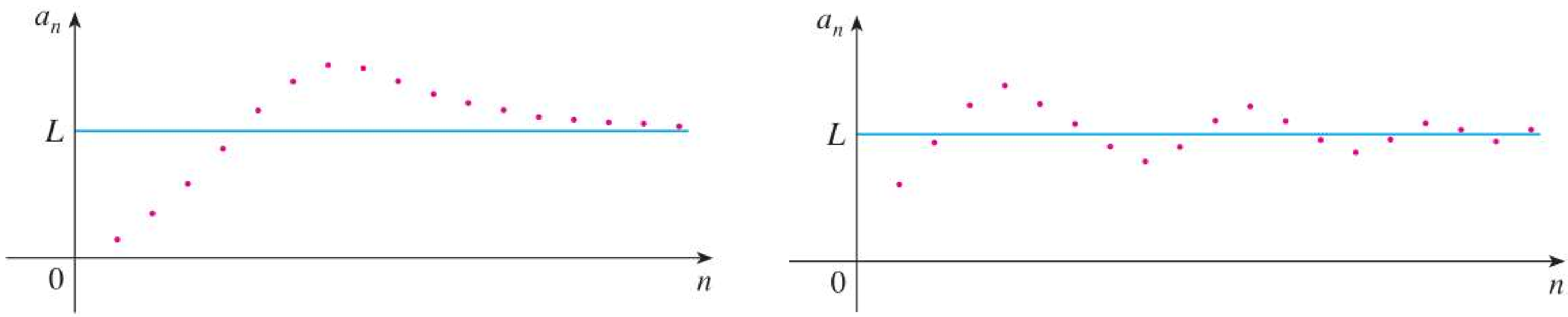
$$(-1)^{n+1} \frac{n+2}{5^n} = a_n$$

# 8.1 Sequences

**1 Definition** A sequence $\{a_n\}$ has the **limit** $L$ and we write

$$\lim_{n \to \infty} a_n = L \qquad \text{or} \qquad a_n \to L \text{ as } n \to \infty$$

if we can make the terms $a_n$ as close to $L$ as we like by taking $n$ sufficiently large. If $\lim_{n \to \infty} a_n$ exists, we say the sequence **converges** (or is **convergent**). Otherwise, we say the sequence **diverges** (or is **divergent**).



# 8.1 Sequences

Find $\lim_{n \to \infty} \frac{n}{n+1}$.

① $\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+\lim_{n\to\infty}\frac{1}{n}} = \frac{1}{1+0} = 1$

② L'Hospital $\lim_{n\to\infty} \frac{n}{n+1} = "\frac{\infty}{\infty}"$

$\overset{L'H}{=} \lim_{n\to\infty} \frac{1}{1} = 1$

Note that sequences are discontinuous so we can't really take derivatives but we can work around that by changing variables.

② $f(x) = \frac{x}{x+1}$ $x \in [0, \infty)$

is differentiable so we can use L'Hospital's rule.

Since $f(n) = a_n$ for $n = 1, 2, 3, \dots$

we conclude that the limit of $f(x)$ is the same as the limit of the sequence.

# 8.1 Sequences

**Applying l'Hospital's Rule to a related function** Calculate $\lim_{n \to \infty} \frac{\ln n}{n}$.

$a_n = \frac{\ln n}{n} \quad n = 1, 2, 3, \ldots$

In order to use theorems that assume differentiability, we formally replace the variable $n$ with a continuous variable $x$.

$f(x) = \frac{\ln x}{x} \quad x \in [1, \infty)$

use L'H Rule for $f(x)$.

$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"}$

$\overset{L'H}{=} \lim_{x \to \infty} \frac{1/x}{1} = \lim_{x \to \infty} \frac{1}{x} = 0.$

Since $\lim_{x \to \infty} f(x) = 0$, $\lim_{n \to \infty} \frac{\ln n}{n}$ is also equal to $0$.

# 8.1 Sequences

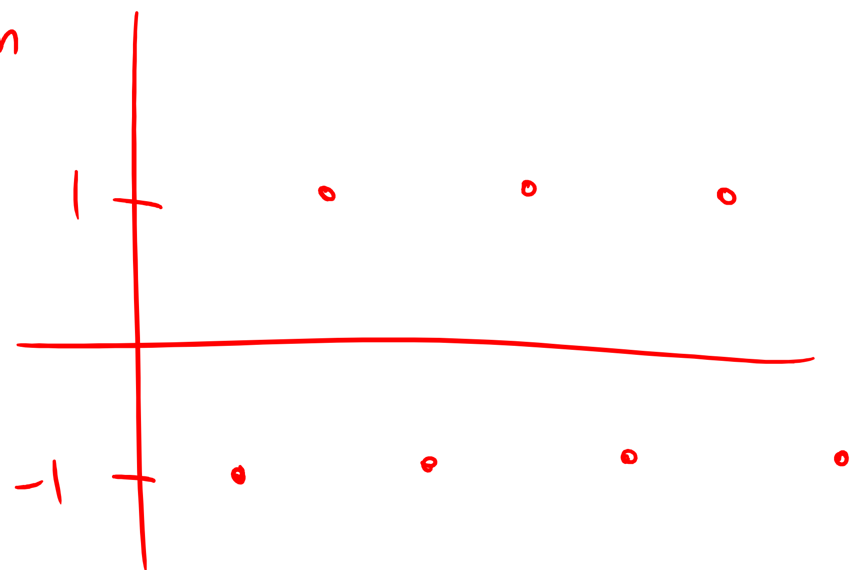
Determine whether the sequence $a_n = (-1)^n$ is convergent or divergent.

Take a look at the expanded form of the sequence.

$a_1, a_2, a_3, a_4, \ldots$

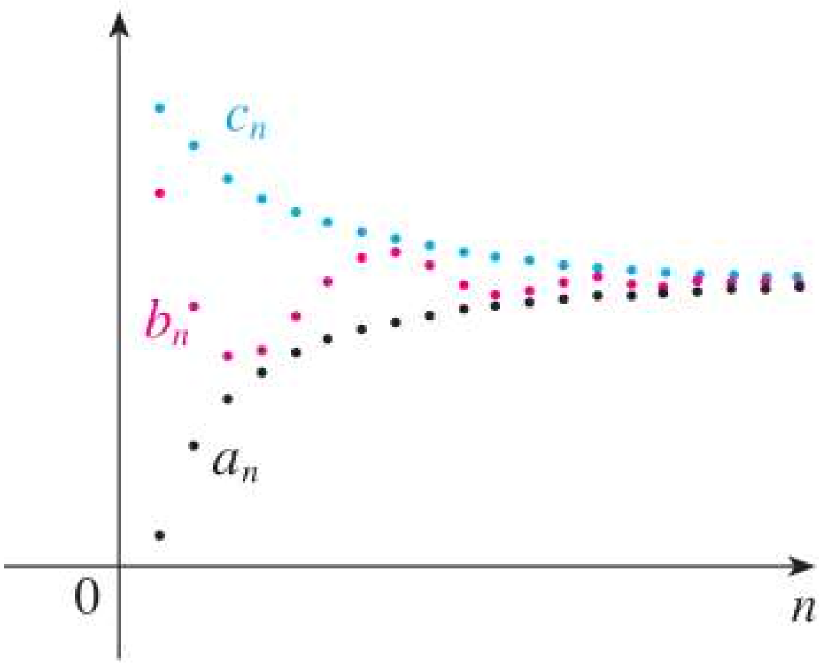
$-1, 1, -1, 1, \ldots$

It is divergent because the values alternate between $-1$ and $1$ so the limit doesn't exist.



# 8.1 The Squeeze Theorem for Sequences

If $a_n \leqslant b_n \leqslant c_n$ for $n \geqslant n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.




Corollary:

If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

# 8.1 Sequences

Evaluate $\lim_{n\to\infty} \frac{(-1)^n}{n}$ if it exists.

Intuition

$n=10$: $\frac{1}{10}$

$n=100$: $\frac{1}{100}$

$n=1000$: $\frac{1}{1000}$

$\frac{1}{\infty}=0$

$n=11$: $\frac{-1}{11}$

$n=101$: $\frac{-1}{101}$

$n=1001$: $\frac{-1}{1001}$

$-\frac{1}{\infty}=0$

Corollary: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

Using the corollary to the squeeze theorem,

$$\lim_{n\to\infty} \left|\frac{(-1)^n}{n}\right| = \lim_{n\to\infty} \frac{1}{n} = 0.$$

so $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$

# 8.1 Geometric Sequences

**EXAMPLE 10** **Limit of a geometric sequence** For what values of $r$ is the sequence $\{r^n\}$ convergent?

we know that exponential sequences like $2^n, e^n$ diverge to infinity. This is because $2, e > 1$. Hence for $r>1$, the sequence $\{r^n\}$ diverges to infinity.

If $r=1$, then the sequence $\{1^n\}$ converges since the expanded form is

$1, 1, 1, 1, \ldots$

for $r=1$ the sequence $\{r^n\}$ converges to 1.

For $0<r<1$, observe that $r^{-1} = \frac{1}{r} > 1$.
Hence the sequence $\frac{1}{r^n} \to \infty$ as $n \to \infty$.
In other words, $r^n \to \frac{1}{\infty} = 0$ as $n \to \infty$.
Therefore for $0<r<1$, the sequence $\{r^n\}$ converges to 0.

Clearly, for $r=0$, the sequence $\{0^n\}$ converges to 0.

For $-1<r<0$, observe that $0<|r|<1$.
Since $|r|^n \to 0$ as $n \to \infty$, by the Corollary to the Squeeze Theorem, $r^n \to 0$ as $n \to \infty$.
Hence for $-1<r<0$, $\{r^n\}$ converges to 0.

# 8.1 Geometric Sequences

**EXAMPLE 10** **Limit of a geometric sequence** For what values of $r$ is the sequence $\{r^n\}$ convergent?

For $r=-1$, the sequence $\{r^n\}$ is divergent since the expanded form is

$-1, 1, -1, 1, -1, 1, \dots$

For $r<-1$, consider two sub-sequences

$\{r^{2k-1}\} = r, r^3, r^5, r^7, \dots$ (odd powers)

$\{r^{2k}\} = r^2, r^4, r^6, r^8, \dots$ (even powers)

Observe that the odd powers of $r$ are strictly negative and the even powers of $r$ are strictly positive.

Since $|r|>1$, $r^{2k} \to \infty$ as $k \to \infty$ while $r^{2k-1} \to -\infty$ as $k \to \infty$.

In other words, the even powers go to $\infty$ while the odd powers go to $-\infty$.

Since the two sub-sequences disagree with their respective limits, the limit doesn't exist for $\{r^n\}$ so it diverges for $r<-1$.

In conclusion, $\{r^n\}$ converges for $r \in (-1, 1]$ while it diverges for $r \in (-\infty, -1] \cup (1, \infty)$.

# 8.1 Geometric Sequences

$\boxed{7}$ The sequence $\{r^n\}$ is convergent if $-1 < r \leqslant 1$ and divergent for all other values of $r$.

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ 1 & \text{if } r = 1 \end{cases}$$