Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Taylor's Inequality

Suppose $T_k(x)$ is a Taylor polynomial centered at a for the function f. Let d be a constant and $|f^{(k+1)}(x)| \leq M$ for values of x satisfying $|x-a| \leq d$. Then for those values of x, the error $R_k(x)$ of the Taylor polynomial $T_k(x)$ satisfies the inequality

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \leq \frac{M}{(k+1)!}d^{k+1}$$

In other words, the error from $T_k(x)$ is bounded by some constants;

$$|R_k(x)| \leq \frac{M}{(k+1)!}d^{k+1}$$

Deciphering Taylor's Inequality:

1. $|x-a| \leq d$ looks **very similar** to the inequality $|x-a| < R$ (R is the radius of convergence.)

2. a is the **center** of the Taylor polynomial, and it is the center of the intervals.

3. d is the **radius of approximation**, which is the distance from the center to the boundary of the **interval of approximation**. In order for the approximation to make sense, d must be less than R:

$$d < R.$$

4. M is computed by **maximizing** $|f^{(k+1)}(x)|$ in the interval of approximation $[a-d, a+d]$. (Usually maximizing an increasing, decreasing, or an oscillating function. Techniques like the Closed Interval Method can be used.)

Controlling the Error

There are three moving parts to Taylor's Inequality:

1. k, the degree of the Taylor polynomial
2. d, the radius of approximation.
3. M, the maximum bound for the $(k+1)$-th derivative of $f(x)$ inside the interval of approximation.

The last moving part M **is dependent on both** k **and** d since the maximum of the $(k+1)$-th derivative is taken over the interval $[a-d, a+d]$.

The **error gets smaller** $(|R_k| \to 0)$ as one either

1. Increases the degree k of the Taylor polynomial $(k \to \infty)$ or
2. Reduces the size of the interval of approximation $(d \to 0)$.

Desmos Examples to Play With

Taylor Polynomials of degree k and the radius of approximation d:

https://www.desmos.com/calculator/bdhuwxcgm7

Graphs of the Taylor polynomials and the errors for various functions:

https://www.cengage.com/math/discipline_content/stewartccc4/2008/14_cengage_tec/publish/deployments/concepts_4e/4c3_tool.html#

Four ways to use Taylor's Inequality - #1

1. Showing that a Taylor series converges to its function f.

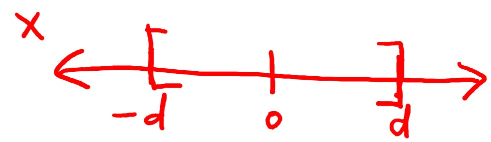
Prove that $f(x) = e^x$ is equal to its Maclaurin series $T(x)$.

$R_k(x)$ measures how different e^x is from $T(x)$. Use Taylor's Inequality to show that for a fixed radius d, $R_k(x) \to 0$ as $k \to \infty$. Observe that $T(x)$ is centered at $a=0$.

Taylor's Inequality

$$|R_k(x)| \leq \frac{M}{(k+1)!} d^{k+1} \quad \text{where} \quad |f^{(k+1)}(x)| \leq M \text{ for } |x-a| \leq d.$$

Since $a=0$, the inequality $|x| \leq d$ gives the interval



$f^{(k+1)}(x) = e^x$ is an increasing function so it is maximized on the right-most x-value in the interval $x=d$.

Therefore $|f^{(k+1)}(x)| = |e^x| = e^x \leq e^d$ for x in $[-d, d]$.

Hence

$$|R_k(x)| \leq \frac{e^d}{(k+1)!} d^{k+1}$$

and taking the limit as k goes to ∞ on both sides of the inequality,

$$\lim_{k\to\infty} |R_k(x)| \le \lim_{k\to\infty} e^d \frac{d^{k+1}}{(k+1)!} = e^d \lim_{k\to\infty} \frac{d^{k+1}}{(k+1)!} = e^d \cdot 0 = 0$$ (since factorials dominate exponentials)

Hence $\lim_{k\to\infty} |R_k(x)| \le 0$ and by the squeeze theorem, $\lim_{k\to\infty} |R_k(x)| = 0$.

Therefore for <u>any</u> fixed radius d, $\lim_{k\to\infty} |R_k(x)| = 0$

which means the difference between e^x and its Taylor series is 0.

In conclusion, for all values of x

$$e^x - T(x) = 0$$

$$e^x = T(x).$$

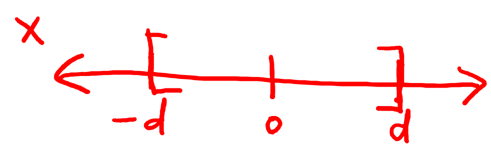
Prove that $f(x) = \sin x$ is equal to its Maclaurin series $T(x)$.

$R_k(x)$ measures how different $\sin x$ is from $T_k(x)$. Use Taylor's Inequality to show that for a fixed radius d, $R_k(x) \to 0$ as $k \to \infty$. Observe that $T(x)$ is centered at $a=0$.

<u>Taylor's Inequality</u>

$$|R_k(x)| \leq \frac{M}{(k+1)!} d^{k+1} \quad \text{where} \quad |f^{(k+1)}(x)| \leq M \text{ for } |x-a| \leq d.$$

Since $a=0$, the inequality $|x| \leq d$ gives the interval



Note that $f^{(k+1)}(x)$ is $\pm \sin x$ or $\pm \cos x$ depending on k.

Since both $\sin x$ and $\cos x$ are bounded above by 1 for all values of x, $|f^{(k+1)}(x)| \leq 1$ for all x. Therefore we set $M=1$.

Hence

$$|R_k(x)| \leq \frac{1}{(k+1)!} d^{k+1}$$ and taking the limit as k goes to ∞ on both sides of the inequality,

$$\lim_{k\to\infty} |R_k(x)| \le \lim_{k\to\infty} \frac{d^{k+1}}{(k+1)!} = 0$$ (since factorials dominate exponentials)

Hence $\lim_{k\to\infty} |R_k(x)| \le 0$ and by the squeeze theorem, $\lim_{k\to\infty} |R_k(x)| = 0$.

Therefore for any fixed radius d, $\lim_{k\to\infty} |R_k(x)| = 0$

which means the difference between $\sin x$ and its Taylor series is 0.

In conclusion, for all values of x

$$\sin x - T(x) = 0$$

$$\sin x = T(x).$$

Four ways to use Taylor's Inequality - #2

2. Determining accuracy; find a bound for the error.

1. Approximate the function $f(x) = \sqrt[3]{x}$ by a Taylor polynomial of degree 2 at $a = 8$.

2. How accurate is this approximation when $7 \le x \le 9$?

① $T_2(x) = \sum_{n=0}^{2} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$

$= f(8) + f'(8)(x-8) + \frac{f''(8)}{2}(x-8)^2$

$f(x) = \sqrt[3]{x} = x^{1/3}$ $\quad f(8) = 2$

$f'(x) = \frac{1}{3}x^{-2/3}$ $\quad f'(8) = \frac{1}{12}$

$f''(x) = -\frac{2}{9}x^{-5/3}$ $\quad f''(8) = -\frac{1}{144}$

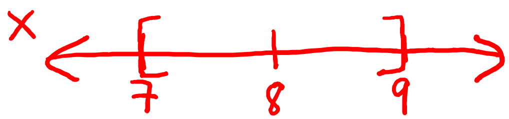
$f(x) \approx T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$

② Use Taylor's Inequality: Suppose $|f^{(k+1)}(x)| \le M$ for values of x satisfying $|x-a| \le d$.

Then

$$|f(x) - T_k(x)| = |R_k(x)| \le \frac{M}{(k+1)!} d^{k+1}$$

For this problem, we are centered at $a=8$ and the interval is $7 \le x \le 9$.



Therefore the distance from the center to the boundary of the interval is $d=1$.

Observe that since we are interested in the accuracy of $T_2(x)$, $k=2$.

Find M: $|f^{(3)}(x)| \le M$ for values of x such that $|x-8| \le 1$ (same as $7 \le x \le 9$).

$f^{(3)}(x) = \frac{10}{27} x^{-8/3}$. Since $x^{-8/3}$ is a decreasing function, $f^{(3)}(x)$ is maximized when $x=7$.

$$|f^{(3)}(x)| = \frac{10}{27} x^{-8/3} \le \frac{10}{27}(7)^{-8/3} = M$$

Putting them all together,

$$|f(x)-T_2(x)| = |R_2(x)| \le \frac{M}{3!} d^3 = \frac{\frac{10}{27}(7)^{-8/3}}{6} 1^3 = 0.000344$$

Therefore when x is between 7 and 9,

$T_2(x)$ is accurate to within 0.000344

Four ways to use Taylor's Inequality - #3

3. The interval of approximation is unknown; solve for the radius of approximation d and use it to find the interval of approximation.

1. What is the maximum error possible in using the approximation

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

when $-0.3 \leq x \leq 0.3$? Use this to approximate $\sin 12°$.

2. For what values of x is this approximation accurate to within 0.00005?

① $f(x) = \sin x$, $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

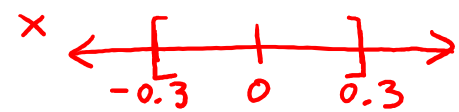
Taylor's Inequality.

$$|f(x) - T_k(x)| \leq \frac{M}{(k+1)!} d^{k+1}$$

where $|f^{(k+1)}(x)| \leq M$ for values of x such that $|x-a| \leq d$.

Since $T_5(x)$ is a Maclaurin polynomial of $\sin x$, the interval is centered at $a=0$.

We are given the interval $-0.3 \leq x \leq 0.3$



The radius of this interval is $d = 0.3$

Since $k=5$, we need to compute $f^{(6)}(x)$ and find a maximum for $-0.3 \leq x \leq 0.3$.

$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f^{(3)}(x) = -\cos x$
$f^{(4)}(x) = \sin x$
$f^{(5)}(x) = \cos x$
$f^{(6)}(x) = -\sin x$

Hence $|f^{(6)}(x)| = |-\sin x| = |\sin x|$. But observe that $|\sin x| \le 1$ for all x so the inequality must also be true for $-0.3 \le x \le 0.3$. Therefore we set $M = 1$.

Putting it all together,

$$|\sin x - T_5(x)| \le \frac{M}{(5+1)!} d^{5+1} = \frac{1}{6!}(0.3)^6 = 0.000001$$

So the maximum error possible in using $T_5(x)$ for $\sin x$ in the interval $-0.3 \le x \le 0.3$ is 0.000001.

$$\sin(12^\circ) = \sin\left(12^\circ \frac{\pi}{180^\circ}\right) = \sin\left(\frac{\pi}{15}\right) \approx T_5\left(\frac{\pi}{15}\right) = \frac{\pi}{15} - \frac{(\pi/15)^3}{3!} + \frac{(\pi/15)^5}{5!}$$

$$= 0.207912$$

② For this problem, we are asked to find the largest interval of x in which $T_5(x)$ is accurate to within 0.00005. Use Taylor's Inequality. (Recall that $a=0$)

$$|\sin x - T_5(x)| \le \frac{M}{(5+1)!} d^{5+1} \quad \text{where} \quad |f^{(6)}(x)| \le M \quad \text{for} \quad |x| \le d.$$

But $f^{(6)}(x) = -\sin x$ so $|f^{(6)}(x)| = |-\sin x| \le 1$ for any x. This means we can use $M=1$ again. Since the bound needs to be less than or equal to 0.00005, we have

$$|\sin x - T_5(x)| \le \frac{1}{6!} d^6 \le 0.00005$$

$$\frac{d^6}{6!} \le 0.00005$$

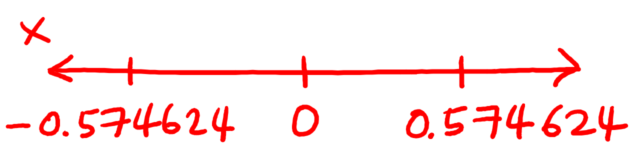
solving for d,

$$d^6 \le 0.036$$

$$d \le 0.574624$$

Therefore the largest value for d is $d = 0.574624$.

Since our interval is centered at $a=0$, we have



In conclusion, for values of x in the interval $[-0.574624, 0.574624]$ $T_5(x)$ is accurate to within 0.00005.

Four ways to use Taylor's Inequality - #4

4. The degree k of the Taylor polynomial needs to be at least some number.

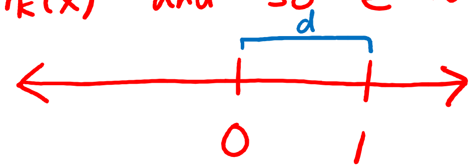
Let $T_k(x)$ be the Taylor polynomial centered at 0 for $f(x) = e^x$. Use Taylor's Inequality to determine the degree k that should be used to estimate the number e^1 with an error less than 0.6.

Taylor's Inequality:

$|f(x) - T_k(x)| \le \frac{M}{(k+1)!} d^{k+1}$ where $|f^{(k+1)}(x)| \le M$ for $|x-a| \le d$.

Centered at 0 means $a=0$. Since $f(x) \approx T_k(x)$, $e^x \approx T_k(x)$ and so $e^1 \approx T_k(1)$.

We need to plug in $x=1$ but we are centered at 0:



For our approximation to be valid up to $x=1$, the radius of approximation d needs to be at least 1; therefore let $d=1$.

The interval of approximation is $[a-d, a+d] = [-1, 1]$.

Since $|f^{(k+1)}(x)| = |e^x| = e^x$ is an increasing function, it is maximized at 1 on the interval $[-1,1]$. Hence $|e^x| \le e^1$ for all x in $[-1,1]$ but we let $M=3$ to avoid setting M equal to the very quantity that we are trying to approximate.

Plugging in the values obtained above and bounding by the given requirement of 0.6, we get

$$|f(1) - T_k(1)| \le \frac{3}{(k+1)!} 1^{k+1} < 0.6$$

$$|e^1 - T_k(1)| \le \frac{3}{(k+1)!} < 0.6$$

Now we must solve for the smallest k that satisfies the above inequality.

$$\frac{3}{(k+1)!} < 0.6$$

$$\frac{3}{0.6} < (k+1)!$$

$$5 < (k+1)!$$

$k=1$

$$(k+1)! = (1+1)! = 2! = 2$$

$k=2$

$$(k+1)! = (2+1)! = 3! = 6$$

$k=2$ is the first integer that satisfies the inequality $5 < (k+1)!$.

Hence the degree of the Taylor polynomial should be at least $k=2$ to estimate e^1 with an error less than 0.6.