Daily Quiz

- Go to Socrative.com and complete the quiz.
- Room Name: HONG5824
- Use your full name.

Approximation Methods for Definite Integrals

When approximating a definite integral $\int_a^b f(x)\,dx$, we rely on integration using power series and apply one of the two methods below:

1. **Integral Test Remainder Estimate**
2. **Alternating Series Remainder Estimate**

(a) Evaluate $\int \frac{1}{1+x^7}dx$ as a power series.

(b) Use part (a) to approximate $\int_0^{0.5} \frac{1}{1+x^7}dx$ correct to within 10^{-7}.

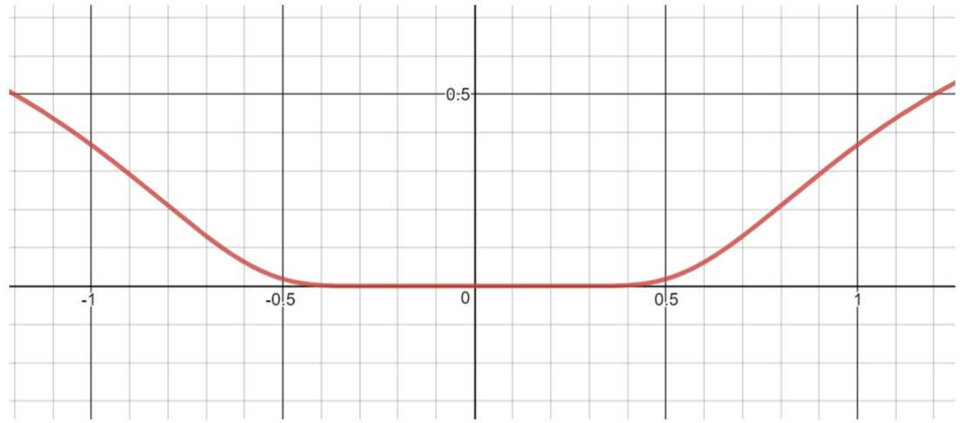
1. Find $\int e^{-x^2}\,dx$ as a power series.

2. Evaluate $\int_0^1 e^{-x^2}\,dx$ correct to within an error of 0.001.

Is a function $f(x)$ really equal to its Taylor series **inside its interval of convergence**?

Not always.

Consider the function



$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

$f(x)$ has derivatives everywhere and $f^{(n)}(0) = 0$ for all n.
But observe that the Taylor series of $f(x)$ centered at 0 is

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = \sum_{n=0}^{\infty} 0 = 0.$$

Because $T(x) = 0$, $T(x)$ converges for all values of x and the interval of convergence is all real numbers $(-\infty, \infty)$.
Does this mean $f(x) = T(x) = 0$ for all real numbers x?

No. $f(x)$ is an example of a function that is **not equal** to its Taylor series inside the interval of convergence.

8.6 When is a function equal to its Taylor series?

To make sure that a function $f(x)$ can be approximated by its Taylor series $T(x)$, we need to compute the **difference** between $f(x)$ and $T(x)$.

Recall the definition of the k**th-degree Taylor polynomial of** $f(x)$ **centered at** a:

$$T_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a)+f'(a)(x-a)+\frac{f''(a)}{2}(x-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(x-a)^k.$$

We defined the **Taylor series** as the limit of the sequence of Taylor polynomials:

$$T(x) = \lim_{k\to\infty} T_k(x)$$

8.6 When is a function equal to its Taylor series?

We say that $f(x)$ is **equal** to its Taylor series if the sequence of Taylor polynomials $T_k(x)$ **converges** to $f(x)$:

$$f(x) = \lim_{k\to\infty} T_k(x)$$

We define $R_k(x) = f(x) - T_k(x)$ as the kth degree **remainder** (or the **error**) of the Taylor series. Then $f(x)$ is equal to its Taylor series if and only if the **remainder (error) vanishes**, i.e.

$$\lim_{k\to\infty} R_k(x) = 0$$

Taylor's Inequality

Suppose $T_k(x)$ is a Taylor polynomial centered at a for the function f. Let d be a constant and $|f^{(k+1)}(x)| \leq M$ for values of x satisfying $|x-a| \leq d$. Then for those values of x, the error $R_k(x)$ of the Taylor polynomial $T_k(x)$ satisfies the inequality

$$|R_k(x)| \leq \frac{M}{(k+1)!}|x-a|^{k+1} \leq \frac{M}{(k+1)!}d^{k+1}$$

In other words, the error from $T_k(x)$ is bounded by some constants;

$$|R_k(x)| \leq \frac{M}{(k+1)!}d^{k+1}$$

Deciphering Taylor's Inequality:

1. $|x-a| \le d$ looks **very similar** to the inequality $|x-a| < R$ (R is the radius of convergence.)

2. a is the **center** of the Taylor polynomial, and it is the center of the intervals.

3. d is the **radius of approximation**, which is the distance from the center to the boundary of the **interval of approximation**. In order for the approximation to make sense, d must be less than R:

$$d < R.$$

4. M is computed by **maximizing** $|f^{(k+1)}(x)|$ in the interval of approximation $[a-d, a+d]$. (Usually maximizing an increasing, decreasing, or an oscillating function. Techniques like the Closed Interval Method can be used.)

Controlling the Error

There are three moving parts to Taylor's Inequality:

1. k, the degree of the Taylor polynomial
2. d, the radius of approximation.
3. M, the maximum bound for the $(k+1)$-th derivative of $f(x)$ inside the interval of approximation.

The last moving part M **is dependent on both** k **and** d since the maximum of the $(k+1)$-th derivative is taken over the interval $[a-d, a+d]$.

The **error gets smaller** ($|R_k| \to 0$) as one either

1. Increases the degree k of the Taylor polynomial ($k \to \infty$) or
2. Reduces the size of the interval of approximation ($d \to 0$).

Desmos Examples to Play With

Taylor Polynomials of degree k and the radius of approximation d:

https://www.desmos.com/calculator/ljbm9jewu0

Graphs of the Taylor polynomials and the errors for various functions:

https://www.cengage.com/math/discipline_content/stewartccc
4/2008/14_cengage_tec/publish/deployments/concepts_4e/4c
3_tool.html#