1. **Derivatives.**

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

Properties

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[c_1 u + c_2 v\Big] = c_1\frac{\mathrm{d}u}{\mathrm{d}x} + c_2\frac{\mathrm{d}v}{\mathrm{d}x} \qquad \text{(linearity)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[u\times v\Big] = \frac{\mathrm{d}u}{\mathrm{d}x}\times v + u\times\frac{\mathrm{d}v}{\mathrm{d}x} \qquad \text{(deriv of a product)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{u}{v}\right] = \frac{\frac{\mathrm{d}u}{\mathrm{d}x}\times v - u\times\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2} \qquad \text{(deriv of a quotient)}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big[f\big(g(x)\big)\Big] = \left[\frac{\mathrm{d}f}{\mathrm{d}g}\times\frac{\mathrm{d}g}{\mathrm{d}x}\right]_{g=g(x)} = f'\big(g(x)\big)\times g'(x) \qquad \text{(chain rule)}$$

$$g(x) = \mathrm{inv}\, f(x) \quad\Rightarrow\quad f\big(g(x)\big) = x \quad\Rightarrow\quad \frac{\mathrm{d}}{\mathrm{d}x}\Big[f\big(g(x)\big)\Big] = 1 \quad\Rightarrow\quad \frac{\mathrm{d}g}{\mathrm{d}x} = \left[\frac{1}{f'(g)}\right]_{g=g(x)}$$

Basic Derivatives

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^a] = a\,x^{a-1} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[e^x] = e^x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\ln x] = \frac{1}{x}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[|x|] = \frac{x}{|x|} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[a^x] = a^x \ln a \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\log_a x] = \frac{1}{x\,\ln a}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\sin x] = \cos x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\tan x] = \sec^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\sec x] = \sec x\,\tan x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\cos x] = -\sin x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\cot x] = -\csc^2 x \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\csc x] = -\csc x\,\cot x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arcsin x] = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\arctan x] = \frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{arcsec}\, x] = \frac{1}{|x|\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[\arccos x] = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{arccot}\, x] = -\frac{1}{1+x^2} \qquad \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{arccsc}\, x] = -\frac{1}{|x|\sqrt{1-x^2}}$$

Higher Order Derivatives

$$\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right] = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}}_{2}[f]$$

$$\frac{\mathrm{d}^3 f}{\mathrm{d}x^3} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}\right] = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}}_{3}[f]$$

$$\vdots \quad = \quad \vdots \quad = \quad \vdots$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{d}^{n-1} f}{\mathrm{d}x^{n-1}}\right] = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}}_{n}[f]$$

2. **Lôpital's Rule.**

- If $f(a) = g(a) = 0$, and f and g are differentiable on an open interval I containing a, and $g'(x) \neq 0$ on I if $x \neq a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

  assuming the limit on the r.h.s. exist.

- If $\lim_{x\to a} f(x) = \pm\infty$, and $\lim_{x\to a} g(x) = \pm\infty$, and and f and g are differentiable on an open interval I containing a, then
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$
assuming the limit on the r.h.s. exist.
- In the limits above $x \to a$ can be replaced by $x \to a^-$, $x \to a^+$, $x \to +\infty$, or $x \to -\infty$.
- Indeterminate form $0 \times \infty$: rewrite it as $0/0$ or ∞/∞.
- Indeterminate form $\infty - \infty$: in general, take common denominator or simplify and factor.
- Indeterminate forms 1^∞, 0^0, and ∞^0: write $y = [f(x)]^{g(x)}$, take logs and use
$$\lim_{x\to a}\ln(y) = \ln\left(\lim_{x\to a} y\right) \quad \Rightarrow \quad \lim_{x\to a} y = \exp\left(\lim_{x\to a}\ln(y)\right).$$

3. **Antiderivatives.**

$$\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x) \quad \Longleftrightarrow \quad F(x) = \int f(x)\,\mathrm{d}x$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\,\mathrm{d}x &= f(x)\\ \int \frac{\mathrm{d}}{\mathrm{d}x}f(x)\,\mathrm{d}x &= f(x) + C\end{aligned}$$

Properties

$$\begin{aligned}\int [c_1 f(x) + c_2 g(x)]\ \mathrm{d}x &= c_1\int f(x)\,\mathrm{d}x + c_2\int g(x)\,\mathrm{d}x && \text{(linearity)}\\ \int f\big(g(x)\big)\,g'(x)\ \mathrm{d}x &= \left[\int f(g)\,\mathrm{d}g\right]_{g=g(x)} && \text{(substitution} \longleftarrow \text{chian rule)}\end{aligned}$$

Basic Antiderivatives

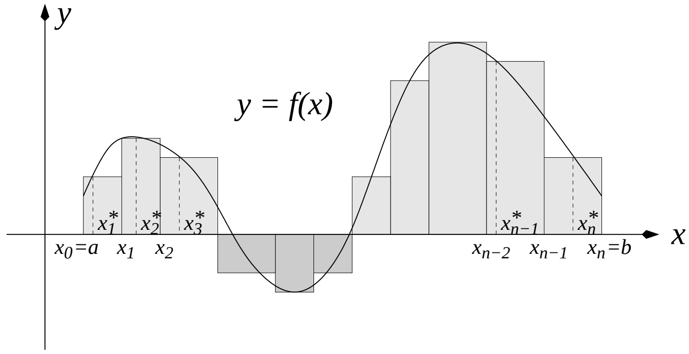
$$\int 0\,\mathrm{d}x = C$$

$$\begin{aligned}\int x^a\,\mathrm{d}x &= \frac{x^{a+1}}{a+1} + C \qquad a \neq -1 & \int e^x\,\mathrm{d}x &= e^x + C\\ \int \frac{1}{x}\,\mathrm{d}x &= \ln|x| + C & \int a^x\,\mathrm{d}x &= \frac{a^x}{\ln a} + C \qquad a \neq 1\end{aligned}$$

$$\begin{aligned}\int \sin x\,\mathrm{d}x &= -\cos x + C & \int \sec^2 x\,\mathrm{d}x &= \tan x + C & \int \sec x\tan x\,\mathrm{d}x &= \sec x + C\\ \int \cos x\,\mathrm{d}x &= \sin x + C & \int \csc^2 x\,\mathrm{d}x &= -\cot x + C & \int \csc x\cot x\,\mathrm{d}x &= -\csc x + C\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x &= \arcsin x + C_+ &&= -\arccos x + C_-\\ \int \frac{1}{1-x^2}\,\mathrm{d}x &= \arctan x + C_+ &&= -\mathrm{arccot}\,x + C_-\\ \int \frac{1}{|x|\sqrt{1-x^2}}\,\mathrm{d}x &= \mathrm{arcsec}\,x + C_+ &&= -\mathrm{arccsc}\,x + C_-\end{aligned}$$

4. **The Definite Integral**



$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\max(\Delta x)_i \to 0} \sum_{k=1}^{n} f(x_k^*)(\Delta x)_k$$

Properties

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

$$\int_a^b f(x)\,\mathrm{d}x = -\int_b^a f(x)\,\mathrm{d}x$$

$$\int_a^b [c_1 f(x) + c_2 g(x)]\,\mathrm{d}x = c_1 \int_a^b f(x)\,\mathrm{d}x + c_2 \int_a^b g(x)\,\mathrm{d}x$$

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$$

$$f(x) \geq g(x) \quad \forall x \in [a,b] \quad \Rightarrow \quad \int_a^b f(x)\,\mathrm{d}x \geq \int_a^b g(x)\,\mathrm{d}x$$

particular cases of this last property

$$g(x) \leq f(x) \leq h(x) \quad \forall x \in [a,b] \quad \Rightarrow \quad \int_a^b g(x)\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b h(x)\,\mathrm{d}x$$

$$\min_{x\in[a,b]} f(x) \leq f(x) \leq \max_{x\in[a,b]} f(x) \quad \Rightarrow \quad (b-a) \times \min_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,\mathrm{d}x \leq (b-a) \times \max_{x\in[a,b]} f(x)$$

The Fundamental Theorem of Calculus

Part 1 If f is continuous on $[a,b]$, then $\displaystyle\int_{t=a}^{x} f(t)\,\mathrm{d}t$ is a continuous function of x on $[a,b]$, differentiable on (a,b), with

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{t=a}^{x} f(t)\,\mathrm{d}t = f(x)\,.$$

Part 2 If f is continuous at every point of $[a,b]$, then

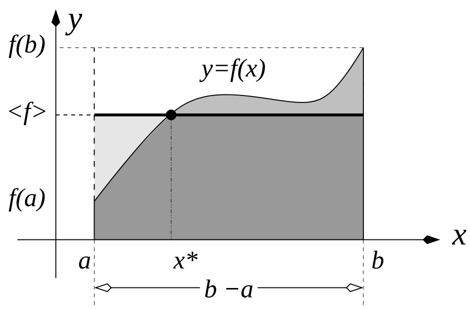
$$\int_{x=a}^{b} f(x)\,\mathrm{d}x = \left[\int f(x)\,\mathrm{d}x\right]_{x=a}^{b}.$$

Definite Integrals and Chain Rule

$$\int_{x=a}^{b} f\big(g(x)\big)g'(x)\,\mathrm{d}x = \int_{g=g(a)}^{g(b)} f(g)\,\mathrm{d}g\,.$$

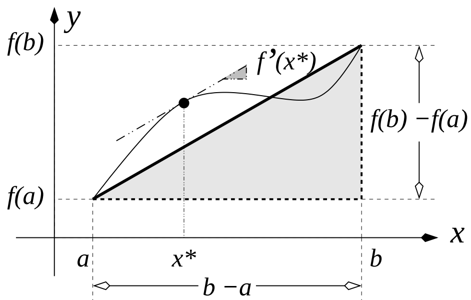
$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{t=u(x)}^{v(x)} f(t)\,\mathrm{d}t = f\big(v(x)\big)\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right) - f\big(u(x)\big)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right).$$

5. **Mean Values.**



If f is continuous on $[a, b]$, then there is at least one point $x^* \in (a, b)$ such that

$$f(x^*) = \frac{1}{b-a} \int_{x=a}^{b} f(x)\,\mathrm{d}x = \langle f \rangle$$



If f is continuous on $[a, b]$ and differentiable on (a, b), then there is at least one point $x^* \in (a, b)$ such that

$$f'(x^*) = \frac{f(b) - f(a)}{b-a} = \langle f' \rangle$$

$$\langle f' \rangle = \frac{1}{b-a} \int_{x=a}^{b} f'(x)\,\mathrm{d}x = \frac{f(b) - f(a)}{b-a}$$