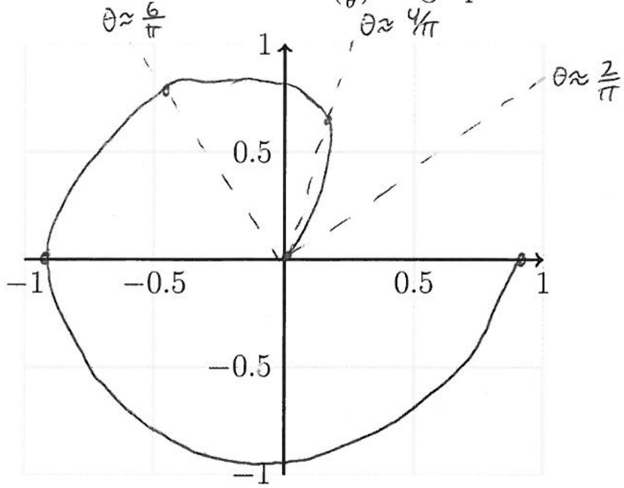
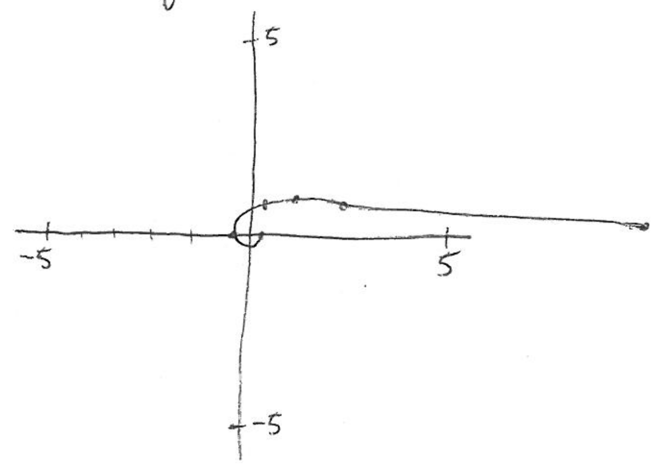
1 Graphing in Polar Coordinates

1. Use the function $r = \cos(\frac{1}{\theta})$ to graph the following:



graph of 2:



(a) When $\theta = \frac{2}{\pi}$, $r = \cos(\frac{\pi}{2}) = 0$

(b) When $\theta = \frac{4}{\pi}$, $r = \cos(\frac{\pi}{4}) = \frac{\sqrt{2}}{2}$

(c) When $\theta = \frac{6}{\pi}$, $r = \cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

(d) When $\theta = \pi$, $r = \cos(\frac{1}{\pi})$, between $\frac{\sqrt{3}}{2}$ and 1

(e) When $\theta = 2\pi$, $r = \cos(\frac{1}{2\pi})$, between $\frac{\sqrt{3}}{2}$ and 1

(f) the arc of the polar function where $\frac{2}{\pi} \leq \theta \leq 2\pi$

2. Use the function $r = \frac{1}{\theta}$ to graph the following:

(a) When $\theta = \frac{1}{10}$, $r = 10$

(b) When $\theta = \frac{\pi}{6}$, $r = \frac{6}{\pi} \approx 2$

(c) When $\theta = \frac{\pi}{4}$, $r = \frac{4}{\pi} \approx \frac{4}{3}$

(d) When $\theta = \frac{\pi}{3}$, $r = \frac{3}{\pi} < 1$

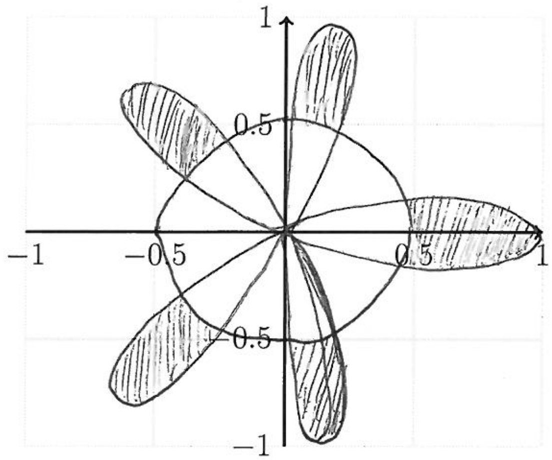
(e) When $\theta = \pi$, $r = \frac{1}{\pi} < \frac{1}{3}$

(f) When $\theta = 2\pi$, $r = \frac{1}{2\pi} < \frac{1}{6}$

(g) the arc of the polar function where $\frac{1}{10} \leq \theta \leq 2\pi$.

2 Identifying Areas

1. Graph the function $r_1 = \cos(5\theta)$.



$\cos(n\theta)$ has n petals when n is odd; 5 is odd, so we have 5 petals

2. Find all values of θ for which $r_1 = 0$. What bounds could you use for θ to set up an integral that will give you the area of 1 petal?

   $\cos(\theta) = 0$ when $\theta = \pi/2, 3\pi/2, 5\pi/2 \ldots$

   so $\cos(5\theta) = 0$ when $\theta = \pi/10, 3\pi/10, 5\pi/10 \ldots$

   The petal on the x-axis lies between $\theta = -\frac{\pi}{10}$ and $\frac{\pi}{10}$

   Is r_1 positive or negative on this region?

   r_1 is positive here (on the first petal)

   r_1 will be negative on the next petal.

3. Suppose you want to find the area inside the petals, but outside the circle $r_2 = \frac{1}{2}$. Find all values of θ for which $r_1 = \frac{1}{2}$.

   $\cos(\theta) = \frac{1}{2}$ when $\theta = -\pi/3, \pi/3, 5\pi/3, \ldots$

   so $\cos(5\theta) = \frac{1}{2}$ when $\theta = \frac{-\pi}{15}, \frac{\pi}{15}, \frac{5\pi}{15}, \ldots$

   note: if r_1 was negative, we would want to find $r_1 = -\frac{1}{2}$ to see where the petal and circle intersect.

   Add the circle $r_2 = \frac{1}{2}$ to the graph above. Then shade in the area you are interested in finding.

   What bounds could you use to set up an integral that will give you the area?

   $\frac{-\pi}{15}$ to $\frac{\pi}{15}$ will give the part of the first petal outside the circle

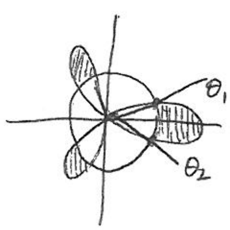
3 Calculating Areas

1. Set up and evaluate the integral that will give you the area swept out by $r = \frac{1}{\theta}$, with $\frac{1}{10} \le \theta \le 2\pi$.

$$\text{area} = \frac{1}{2}\int_a^b r^2\, d\theta$$

$$= \frac{1}{2}\int_{\frac{1}{10}}^{2\pi} \frac{1}{\theta^2}\, d\theta = \frac{1}{2}\left(\frac{-1}{\theta}\right)\Big|_{1/10}^{2\pi} = \frac{-1}{4\pi} + \frac{10}{2} = 5 - \frac{1}{4\pi} \approx 4.92$$

2. Set up, but do not evaluate, the integral that will give you the area inside $r = \cos(3\theta)$ and outside the circle $r = \frac{1}{2}$.



$$\cos(3\theta_1) = \frac{1}{2}$$
$$3\theta_1 = \frac{\pi}{3}$$
$$\theta_1 = \frac{\pi}{9}$$
$$\theta_2 = -\frac{\pi}{9}$$

area outside circle in 1 petal:

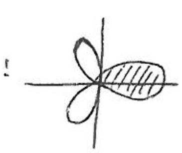
$$\frac{1}{2}\int_{-\pi/9}^{\pi/9} \cos^2(3\theta)\, d\theta - \frac{1}{2}\int_{-\pi/9}^{\pi/9} (1/2)^2\, d\theta$$

total area (in 3 petals:

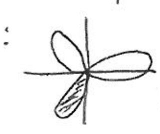
$$\frac{3}{2}\int_{-\pi/9}^{\pi/9} \cos^2(3\theta)\, d\theta - \frac{3}{2}\int_{-\pi/9}^{\pi/9} (1/2)^2\, d\theta$$

3. Use a calculator to find $\frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)d\theta$ and $\frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2(3\theta)d\theta$. What do these integrals represent? Explain why you get the values you do.

$$\frac{1}{2}\int_{-\pi/6}^{\pi/6} \cos^2(3\theta)\, d\theta = \frac{\pi}{12} \approx .262$$ This is the area of :



$$\frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2(3\theta)\, d\theta = \frac{\pi}{12} \approx .262$$ This is the area of :



The petal is opposite of the range of θ we have, since on this range, r is negative. However area is calculated with r^2, and is never negative

4 Calculating Arclength

1. Set up, but do not evaluate, the integral that will give you arclength of $r = \frac{1}{\theta}$, with $\frac{1}{10} \leq \theta \leq 2\pi$.

$x = \frac{1}{\theta}\cos\theta \qquad y = \frac{1}{\theta}\sin\theta$

$\frac{dx}{d\theta} = -\frac{1}{\theta^2}\cos\theta - \frac{1}{\theta}\sin\theta \qquad \frac{dy}{d\theta} = -\frac{1}{\theta^2}\sin\theta + \frac{1}{\theta}\cos\theta$

$$\text{arclength} = \int_a^b \sqrt{(x')^2+(y')^2}\, d\theta = \int_{1/10}^{2\pi} \sqrt{\frac{\cos^2\theta}{\theta^4} + \frac{\sin^2\theta}{\theta^2} + \frac{\sin\theta\cos\theta}{\theta^2} + \frac{\sin^2\theta}{\theta^4} + \frac{\cos^2\theta}{\theta^2} - \frac{\sin^2\theta\cos^2\theta}{\theta^2}}\, d\theta$$

$$= \int_{1/10}^{2\pi} \sqrt{\frac{1}{\theta^4} + \frac{1}{\theta^2}}\, d\theta$$

2. Find the value of the previous integral using a calculator.

$$\int_{1/10}^{2\pi} \sqrt{\frac{1}{\theta^4} + \frac{1}{\theta^2}}\, d\theta \approx 11.4748$$

3. Set up and evaluate the integral that will give you the arclength of $r = \cos(\theta)$, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$. From $\frac{\pi}{2}$ to $\frac{3\pi}{2}$? From 0 to 2π? Interpret the results.

$x = (\cos\theta)^2$

$y = \cos\theta \cdot \sin\theta$

$\frac{dx}{d\theta} = 2\cos\theta(-\sin\theta)$

$\frac{dy}{d\theta} = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$

$$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = 4\cos^2\theta\sin^2\theta + (2\cos^2\theta - 1)^2$$
$$= 4\cos^2\theta\sin^2\theta + 4\cos^4\theta - 4\cos^2\theta + 1$$
$$= 4\cos^2\theta(\sin^2\theta + \cos^2\theta) - 4\cos^2\theta + 1$$
$$= 1$$

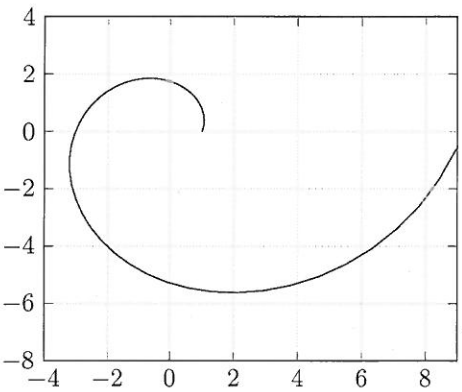
arclength: $\int_{-\pi/2}^{\pi/2} 1\, d\theta = \theta\Big|_{-\pi/2}^{\pi/2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi =$ circumference of circle with radius $\frac{1}{2}$

$\int_{\pi/2}^{3\pi/2} 1\, d\theta = \frac{3\pi}{2} - \frac{\pi}{2} = \pi =$ circumference of circle with radius $\frac{1}{2}$

$\int_0^{2\pi} 1\, d\theta = 2\pi =$ 2 × the circumference of a circle with radius $\frac{1}{2}$

$r = \cos(\theta)$ is a circle with ~~circumference~~ radius $\frac{1}{2}$. When θ goes from 0 to 2π, we trace around the circle twice (the second time, r is negative)

4. Find the arclength of the following curve, from $t = 0$ to $t = \ln 6$.



$$x(t) = e^t \cos(\sqrt{8}t)$$
$$y(t) = e^t \sin(\sqrt{8}t)$$

$$x'(t) = e^t \cos\sqrt{8}t - \sqrt{8}e^t \sin\sqrt{8}t$$
$$y'(t) = e^t \sin\sqrt{8}t + \sqrt{8}e^t \cos\sqrt{8}t$$

$$(x')^2 + (y')^2 = e^{2t}\cos^2\sqrt{8}t + 8e^{2t}\sin^2\sqrt{8}t - \sqrt{8}e^{2t}\cos\sqrt{8}t\sin\sqrt{8}t$$
$$+ e^{2t}\sin^2\sqrt{8}t + 8e^{2t}\cancel{\sin^2}\cos^2\sqrt{8}t + \sqrt{8}e^{2t}\sin\sqrt{8}t\cos\sqrt{8}t$$
$$= e^{2t} + 8e^{2t} = 9e^{2t}$$

$$\text{arclength} = \int_0^{\ln(6)} \sqrt{9e^{2t}}\,dt = \int_0^{\ln(6)} 3e^t\,dt$$
$$= 3e^t \Big|_0^{\ln(6)}$$
$$= 3e^{\ln(6)} - 3e^0$$
$$= 3 \cdot 6 - 3 = 15$$