1 Sequences and Series

1. Use $a_n = \frac{2}{n+1}$ for the following questions.

   (a) Write the sequence made up of the given terms. Calculate the first 3 terms of the sequence.

   sequence: $\frac{2}{2}, \frac{2}{3}, \frac{2}{4}, \ldots$

   (b) Write the series made up of the given terms. Calculate the first 3 partial sums.

   series: $\frac{2}{2} + \frac{2}{3} + \frac{2}{4} + \ldots$

   $S_1 = 1 \qquad S_2 = 1 + \frac{2}{3} \qquad S_3 = 1 + \frac{2}{3} + \frac{1}{2}$

2. Does the sequence you wrote above converge? If so, to what?

   yes, the sequence converges to 0, since

   $$\lim_{n\to\infty} \frac{2}{n+1} = 0$$

3. Does the series you wrote above converge?

   the series does not converge

4. Describe the difference between a sequence and a series.

   a sequence is a list of terms, and a series is all of the terms in a sequence added together

2 Geometric Series

1. Which of the following are geometric series? How can you tell?

   In a geometric series, the ratio of consecutive terms is constant

   ✓(a) $\sum_{n=0}^{\infty} 3(\frac{3}{4})^n$

   ✓(b) $\sum_{n=0}^{\infty} 3(\frac{3}{4})^{2n}$

   ✓(c) $\sum_{n=0}^{\infty} 3(\frac{4}{3})^{n+1}$

   ✗(d) $\sum_{n=0}^{\infty} 3(\frac{1}{4})^{n^2} \rightarrow \frac{a_2}{a_1} \neq \frac{a_3}{a_2}$

   ✓(e) $6 + 3 + 1.5 + .75 + \cdots \rightarrow \frac{3}{6} = \frac{1.5}{3} = \frac{.75}{1.5} \cdots$

   ✗(f) $-16 + 9 - 4 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{9} + \cdots \rightarrow \frac{9}{-16} \neq \frac{-4}{9}$

   ✓(g) $12 - 4 + \frac{4}{3} - \frac{4}{9} + \cdots$

2. $\sum_{n=0}^{\infty} 7(\frac{2}{3})^n$ is a geometric series.

   (a) Write down (expand) the first few partial sums of the given series.

   $S_1 = 7 \quad S_2 = 7 + 7(\frac{2}{3}) \quad S_3 = 7 + 7(\frac{2}{3}) + 7(\frac{2}{3})^2 = \frac{7 - 7(\frac{2}{3})^2}{1 - (\frac{2}{3})}$

   (b) What is the n^{th} partial sum of the given series?

   $S_n = \frac{7 - 7(\frac{2}{3})^n}{1 - (\frac{2}{3})}$ note that in S_3, the exponent of $(\frac{2}{3})$ in the numerator was, which matches our ~~formula~~ formula here

   (c) What does the given series converge to?

   $a = 7, \; r = \frac{2}{3}. \quad \frac{a}{1-r} = \frac{7}{1-\frac{2}{3}} = \frac{7}{1/3} = 21$

3. $\sum_{n=3}^{\infty} 7(\frac{2}{3})^{2n}$ is a geometric series.

   (a) Write down (expand) the first few partial sums of the given series.

   $S_1 = 7(\frac{2}{3})^6 \quad S_2 = 7(\frac{2}{3})^6 + 7(\frac{2}{3})^8 = \frac{7(\frac{2}{3})^6 - 7(\frac{2}{3})^{10}}{1 - (\frac{2}{3})^2} \quad S_3 = 7(\frac{2}{3})^6 + 7(\frac{2}{3})^8 + 7(\frac{2}{3})^{10} = \frac{7(\frac{2}{3})^6 - 7(\frac{2}{3})^{12}}{1 - (\frac{2}{3})^2}$

   (b) What is the n^{th} partial sum of the given series?

   $S_n = \frac{7(\frac{2}{3})^6 - 7(\frac{2}{3})^{2(n+3)}}{1 - (\frac{2}{3})^2}$ note that when $n = 2, 3$, this formula gives the correct exponent for $(\frac{2}{3})$ in the numerator

   (c) What does the given series converge to?

   $a = 7(\frac{2}{3})^6, \; r = (\frac{2}{3})^2, \quad \frac{a}{1-r} = \frac{7(\frac{2}{3})^6}{1 - (\frac{2}{3})^2}$

3 Integral Comparison

If possible, use the n^{th} term (divergence) test, the integral comparison test, or the p-series test to determine whether the following series converge or diverge. State which test you used, and if none of them apply, explain why.

1. $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$ Using $u = \ln(x)$, $\int \frac{\ln(x)}{x}\,dx = \frac{\ln(x)^2}{2} + C$
   So $\int_1^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{b\to\infty} \frac{\ln(x)^2}{2}\Big|_1^b$, $\lim_{b\to\infty} \frac{\ln(b)^2}{2}$ diverges, so the series diverges

2. $\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^2}$ Using $u = \ln(x)$, $\int \frac{1}{x(\ln(x))^2}\,dx = \frac{-1}{\ln(x)} + C$
   So $\int_2^{\infty} \frac{1}{x(\ln(x))^2}\,dx = \lim_{b\to\infty} \frac{-1}{\ln(x)}\Big|_2^b = \lim_{b\to\infty} \frac{-1}{\ln(b)} + \frac{1}{\ln(2)} = \frac{1}{\ln(2)}$
   The integral converges, so the series converges as well.

3. $\sum_{n=2}^{\infty} \frac{n}{\ln(n)}$ $\lim_{n\to\infty} \frac{n}{\ln(n)} \neq 0$, so the series diverges by the n^{th} term test

4. $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $a_n = \frac{1}{n^p}$, with $p=2$ $p>1$, so the series converges by the p-test

5. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$ $\lim_{n\to\infty} \frac{\sin(n)}{n^2} = 0$, so the n^{th} term test doesn't apply
   $f(x) = \frac{\sin(x)}{x^2}$ is not a decreasing function, so the integral test does not apply
   $a_n \neq \frac{1}{n^p}$, so the p-test does not apply

6. $\sum_{n=1}^{\infty} e^{-n}$ $\int e^{-x}\,dx = -e^{-x} + C$, so $\int_1^{\infty} e^{-x}\,dx = \lim_{b\to\infty} -e^{-x}\Big|_1^b = \lim_{b\to\infty} -e^{-b} + e^{-1} = e^{-1}$
   The integral converges, so the series converges as well.

7. $\sum_{n=1}^{\infty} ne^{-n}$ Using Integration by Parts, $\int xe^{-x} = -xe^{-x} - e^{-x} + C$
   So $\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty} -xe^{-x} - e^{-x}\Big|_1^b = \lim_{b\to\infty} -be^{-b} - e^{-b} + e^{-1} + e^{-1} = 2e^{-1}$
   The integral converges, so the series converges as well.

8. $\sum_{n=1}^{\infty} \frac{e^n}{n}$ $\lim_{n\to\infty} \frac{e^n}{n} \neq 0$, so the series diverges by the n^{th} term test

4 Comparison Tests

For each of the following series, try to determine if the series converges or diverges. For practice, try both the term-size comparison test and the limit comparison test to see if one or both or neither works, explaining why.

1. $\sum_{n=1}^{\infty} \frac{1}{n+\ln(n)}$

limit comparison to $\frac{1}{n}$ tells us that since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{n+\ln(n)}$ diverges too

term comparison to $\frac{1}{n}$ tells us nothing, since the inequality goes the wrong way. Term comparison to $\frac{1}{2n}$ tells us our series diverges

2. $\sum_{n=1}^{\infty} \frac{2}{n(\ln(n))^2}$

There is no obvious comparison (term or limit) to help us here. This is a case for integral comparison!

3. $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^7-n^3-4}}$

$\frac{n^2}{\sqrt{n^7}} \leq \frac{n^2+1}{\sqrt{n^7-n^3-4}}$, and since $\sum \frac{n^2}{\sqrt{n^7}} = \sum \frac{1}{n^{1.5}}$ converges, this comparison does not help

limit comparison: $\lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^7-n^3-4}} \cdot \frac{\sqrt{n^7}}{n^2} = 1$, so since $\sum \frac{1}{n^{1.5}}$ converges, the series converges

4. $\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^6-n^3-4}}$

$\frac{n^2}{\sqrt{n^6}} \leq \frac{n^2+1}{\sqrt{n^6-n^3-4}}$, and since $\sum \frac{n^2}{\sqrt{n^6}} = \sum \frac{1}{n}$ diverges, term comparison tells us the series diverges

limit comparison: $\lim_{n\to\infty} \frac{n^2+1}{\sqrt{n^6-n^3-4}} \cdot \frac{\sqrt{n^6}}{n^2} = 1$, so since $\sum \frac{1}{n}$ diverges, the series diverges

5. $\sum_{n=1}^{\infty} \frac{1}{n^2-n}$

$0 \leq \frac{1}{n^2} \leq \frac{1}{n^2-n}$ and since $\sum \frac{1}{n^2}$ converges, this comparison doesn't help us

limit comparison: $\lim_{n\to\infty} \frac{1}{n^2-1} \cdot \frac{n^2}{1} = 1$, so since $\sum \frac{1}{n^2}$ converges, the series converges

6. $\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$

Intuitively, we expect $\sum \frac{\sin(n)}{n^2}$ to behave like $\sum \frac{1}{n^2}$. However, term comparison requires $0 \leq a_n \leq b_n$, so it does not apply. $\lim_{n\to\infty} \frac{\sin(n)}{n^2} \cdot \frac{n^2}{1} = \lim_{n\to\infty} \sin(n)$, which diverges, so the limit comparison doesn't help either

7. $\sum_{n=1}^{\infty} e^{-(n+1)}$

Both limit comparison and term comparison to e^{-n} tell us the series converges. This could also be shown using integral comparison

8. $\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n}$

$0 \leq \frac{1}{n} \leq \frac{(\ln(n))^2}{n}$, and since $\sum \frac{1}{n}$ diverges, the series diverges

limit comparison: $\lim_{n\to\infty} \frac{(\ln(n))^2}{n} \cdot \frac{n}{1} = \lim_{n\to\infty} \ln(n)^2$, which diverges, so this test does not help us.