MATH 2300 – review problems for Exam 3, part 1

1. Find the radius of convergence and interval of convergence for each of these power series:

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x+5)^n}{2^n \ln n}$

   Solution: Strategy: use the ratio test to determine that the radius of convergence is 2, so the endpoints are $x = -7$ and $x = -3$. At $x = -7$, we have the series $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}$, use alternating series test (don't forget to show hypotheses are met)to show that this series converges. At $x = -3$ we have $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln n}$, use term-size comparison test, comparing to $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n}$ to show the series diverges. Interval of convergence is $[-7, -3)$.

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x-1)^n}{4^n}$

   Solution: Strategy: use the ratio test to determine that the radius of convergence is 4, so the endpoints are $x = -3$ and $x = 5$. At $x = -3$ we have the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n n$, which we can show diverges by the divergence test. At $x = 5$ we have the series $\displaystyle\sum_{n=0}^{\infty} n$, which we can also show diverges by the divergence test. The interval of convergence is $(-3, 5)$.

   (c) $\displaystyle\sum_{n=0}^{\infty} n!(3x+1)^n$

   Solution: Strategy: Use the ratio test to determine the radius of convergence.

$$\lim_{n\to\infty} \frac{(n+1)!(3x+1)^n}{n!(3x+1)^{n+1}} = \lim_{n\to\infty} \frac{n+1}{3x+1} = \infty$$

   (provided $3x + 1 \neq 0$). So the radius of convergence is 0. The only "endpoint" is when $3x + 1 = 0$, or $x = -\frac{1}{3}$. At this point, the sum becomes $\sum_{n=0}^{\infty} 0$, which converges. So the interval of convergence is actually just a point, $x = -\frac{1}{3}$.

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1} x^n}{n^3+1}$

   Solution: Strategy: use the ratio test to show the radius of convergence is $\frac{1}{2}$, so the endpoints are $x = -\frac{1}{2}$ and $x = \frac{1}{2}$. At $x = -\frac{1}{2}$ we have the series $\displaystyle -2\sum_{n=0}^{\infty} \frac{1}{n^3+1}$. You can show this converges using term-size comparison, comparing to $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$. At $x = \frac{1}{2}$ we have the series $\displaystyle -2\sum_{n=0}^{\infty} \frac{(-1)^n}{n^3+1}$, which we can show converges absolutely by the term-size comparison test. The interval of convergence is $[-1/2, 1/2]$.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n x^n}{n!}$

Solution: Again, use the ratio test.

$$\lim_{n\to\infty} \frac{\ln(n+1)\cdot x^{n+1}\cdot n!}{\ln n\cdot x^n\cdot(n+1)!} = |x|\cdot \lim_{n\to\infty}\frac{1}{n+1}\cdot\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n}.$$

Use L'Hopital's rule on the last limit, we get a limit of $|x|\cdot 0\cdot 1 = 0$, regardless of the value of x. So the radius of convergence is infinite, and the interval of convergence is $(-\infty,\infty)$ (meaning that the series converges for all x).

2. Let

$$f(x) = \sum_{n=1}^{\infty}\frac{(x+4)^n}{n^2}$$

Find the intervals of convergence of f and f'. For f: $[-5,-3]$. For f': $[-5,-3)$.

3. If $\sum b_n(x-2)^n$ converges at $x=0$ but diverges at $x=7$, what is the largest possible interval of convergence of this series? What's the smallest possible? Largest: $[-3,7)$. Smallest: $[0,4)$.

4. The power series $\sum c_n(x-5)^n$ converges at $x=3$ and diverges at $x=11$. What are the possibilities for the radius of convergence? What can you say about the convergence of $\sum c_n$? Can you determine if the series converges at $x=6$? At $x=7$? At $x=8$? at $x=2$? At $x=-1$? At $x=-2$? At $x=12$? At $x=-3$? The radius of convergence must be between 2 and 6 (inclusive). When we substitute $x=6$, we get $\sum c_n$, which must converge since $x=6$ is inside the radius of convergence. The series converges at $x=6$. The series diverges at $x=-2$. We don't have enough information to determine convergence at $x=2$ or $x=8$. We also can't determine convergence at $x=-1$ or $x=7$, which possibly lie right at the edge of the interval of convergence.

5. The series $\sum c_n(x+2)^n$ converges at $x=-4$ and diverges at $x=0$. What can you say about the radius of convergence of the power series? What can you say about the convergence of $\sum c_n$? What can you say about the convergence of the series $\sum c_n 2^n$? What can you say about the convergence/divergence of the series at $x=-1$? At $x=-3$? At $x=1$? At $x=-10$? This time the radius of convergence must be exactly 2, so the interval of convergence is $[-4,0)$ When we substitute $x=-1$ we get $\sum c_n$, which must converge since $x=-1$ is within the interval of convergence. When we substitute $x=0$ we get $\sum c_n 2^n$, which we have been told diverges. The series converges at $x=-1$ and at $x=-3$ and diverges at $x=1$ and $x=10$.

6. Say that $f(x)=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}$. Find $f'(x)$ by differentiating termwise.
$f'(x)=\sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!}=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$. Note that $f(x)=\sin x$, and $f'(x)=\cos x$.

7. Use any method to find a power series representation of each of these functions, centered about $a=0$. Give the interval of convergence (Note: you should be able to give this interval based on your derivation of the series, not by using the ratio test.)

(a) $\displaystyle\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$

(b) $\dfrac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}$

(c) $\arctan x = \sum_{n=0}^{\infty} \dfrac{(-1)^n x^{2n+1}}{(2n+1)}$

(d) $xe^x - x = \sum_{n=1}^{\infty} \dfrac{x^{n+1}}{n!}$

(e) $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{x^n}{n}$

(f) $x\ln(1+3x^2) = \sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{3^n x^{2n+1}}{n}$

(g) $\dfrac{\sin(-2x^2)}{x} = \sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{2^{2n+1}x^{4n+1}}{(2n+1)!}$

(h) $\dfrac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$

(i) $\displaystyle\int \frac{1}{1+x^5}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n x^{5n+1}}{5n+1} + C$

8. Determine the function or number represented by the following series:

(a) $\sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$

(b) $\sum_{n=1}^{\infty} n x^{n} = \frac{x}{(1-x)^2}$

(c) $\sum_{n=0}^{\infty} \dfrac{x^{2n}}{5^{2n} n!} = e^{x^2/25}$

(d) $\sum_{n=0}^{\infty} \dfrac{(-1)^n 2^{2n} x^{2n+1}}{(2n+1)!} = \frac{1}{2}\sin 2x$

(e) $\sum_{n=1}^{\infty} \dfrac{x^{2n}}{n} = -\ln(1-x^2)$

(f) $\sum_{n=0}^{\infty} \dfrac{(-1)^n 3^{2n}}{(2n)!} = \cos(3)$

9. A car is moving with speed 20 m/s and acceleration 2 m/s^2 at a given instant. Using a second degree Taylor polynomial, estimate how far the car moves in the next second.

Solution: You should use the Taylor polynomial $P_2(x) = x^2 + 20x + C$ where C is some constant. Then the best estimation for how far the car moves in the next second is

$$P_2(1) - P_2(0) = 21 + C - C = 21 \text{ meters}$$

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10. Estimate $\int_0^1 \frac{\sin t}{t} dt$ using a 3rd degree Taylor Polynomial. What degree Taylor Polynomial should be used to get an estimate within 0.005 of the true value of the integral? (Hint: use the alternating series estimate). Answer: $\frac{17}{18}$. The value is estimated by the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!}$, the nth term is less than 0.005 when $n = 2$, so we must add only two terms, $1 - \frac{1}{3 \cdot 3!}$

11. Calculate the Taylor series of $\ln(1 + x)$ by two methods. First calculate it "from scratch" by finding terms from the general form of Taylor series. Then calculate it again by starting with the Taylor series for $f(x) = \frac{1}{1-x}$ and manipulating it. Determine the interval of convergence each time.

12. Express the integral as an infinite series.

$$\int \frac{e^x - 1}{x} \, dx$$

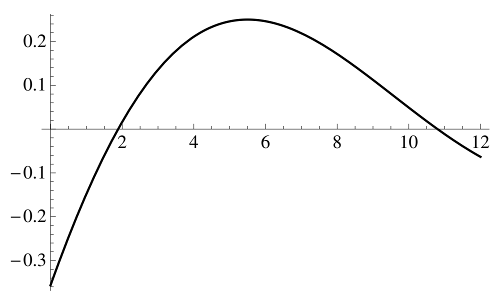
$\int \frac{e^x-1}{x} \, dx = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!} + C$

13. Let $f(x) = \dfrac{1}{1 - x}$.

   (a) Find an upper bound M for $|f^{(n+1)}(x)|$ on the interval $(-1/2, 1/2)$. $2^{n+2} \cdot (n + 1)!$

   (b) Use this result to show that the Taylor series for $\dfrac{1}{1-x}$ converges to $\dfrac{1}{1-x}$ on the interval $(-1/2, 1/2)$.By part (a) and by Taylor's inequality, we have $|R_n(x)| \leq \frac{2^{n+2} \cdot (n+1)!}{(n+1)!} |x|^{n+1} = 2 \cdot (2|x|)^{n+1}$ on $(-1/2, 1, 2)$. But the fact that $|x| < 1/2$ on this interval tells us that $\lim_{n\to\infty} |R_n(x)| \leq 2 \cdot \lim_{n\to\infty} (2|x|)^{n+1} = 0$ on this interval. But remember that $R_n(x) = f(x) - P_n(x)$, where $P_n(x)$ is the nth degree Taylor polynomial for $f(x)$. So $P_n(x) \to f(x)$ as $x \to \infty$, and we're done.

14. Consider the function $y = f(x)$ sketched below.



Suppose $f(x)$ has Taylor series

$$f(x) = a_0 + a_1(x - 4) + a_2(x - 4)^2 + a_3(x - 4)^3 + ...$$

about $x = 4$.

   (a) Is a_0 positive or negative? Please explain.$a_0 > 0$, because the function is positive at $x = 4$.

   (b) Is a_1 positive or negative? Please explain. $a_1 > 0$, because the function is increasing at $x = 4$.

   (c) Is a_2 positive or negative? Please explain. $a_2 < 0$, because the function is concave down at $x = 4$.

15. How many terms of the Taylor series for $\ln(1+x)$ centered at $x=0$ do you need to estimate the value of $\ln(1.4)$ to three decimal places (that is, to within .0005)? We will use the error bound. The error bound corresponding to $P_n(0.4)$ is given by $\dfrac{M(0.4)^{n+1}}{(n+1)!}$, where M is the maximum of $|f^{n+1}(u)|$ on the interval $[0, 0.4]$. For $n \geq 1$, the derivatives of $f(x) = \ln(1+x)$ are given by the following formula: $f^n(x) = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$ Clearly, $|f^{n+1}(u)| = \dfrac{n!}{(1+u)^{n+1}}$ is decreasing on the interval $[0, 0.4]$, so $M = \dfrac{n!}{(1+0)^{n+1}} = n!$ The error bound is then $\dfrac{n!(0.4)^{n+1}}{(n+1)!} = \dfrac{(0.4)^{n+1}}{n+1}$. The first n for which the error bound is smaller than 0.0005 is $n = 6$. (Note: sticking strictly to the method of the textbook, you would find the maximum of $|f^{n+1}(u)|$ on the interval $[-0.4, 0.4]$. In this method, substitute $x = -0.4$ to find the bound M.)

16. (a) Find the 4th degree Taylor Polynomial for $\cos x$ centered at $a = \pi/2$.
    $P_4(x) = -(x - \frac{\pi}{2}) + \frac{1}{6}(x - \frac{\pi}{2})^3$

    (b) Use it to estimate $\cos(89°)$. $89° = \frac{89\pi}{180}$, so $\cos(89°) \approx -(-\frac{\pi}{180}) + \frac{1}{6}(-\frac{\pi}{180})^3 \approx 0.0174524064$

    (c) Use Taylor's inequality to determine what degree Taylor Polynomial should be used to guarantee the estimate to within .005. The $(n+1)$st derivative of $\cos(x)$ is $\pm \sin x$ or $\pm \cos x$, so an upper bound for $f^{(n+1)}(x)$ is $M = 1$. $\left|E_n(\frac{89\pi}{180})\right| \leq \frac{1}{(n+1)!}\left|\frac{89\pi}{180} - \frac{90\pi}{180}\right|^{n+1}$. When $n = 1$ this quantity is $< .005$, so the first term gives an approximation to within 0.005.

17. (a) Find the 3rd degree Taylor Polynomial $P_3(x)$ for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.
    Solution:

$$P_3(x) = 1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{16}$$

    (b) Use the Taylor Polynomial in part (a) to estimate $\sqrt{1.1}$.
    Solution:

$$\sqrt{1.1} \approx P_3(1.1) = 1.0488125$$

    (c) Use Taylor's inequality to determine how accurate is your estimate is guaranteed to be.
    Solution: $|f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$. This is a decreasing function on the interval $[1, 1.1]$, so its largest value occurs at $x = 1$. Thus I can use $f(1) = \frac{15}{16}$ for M. By Taylor's inequality, the absolute value of my error is bounded by $\frac{M}{4!}(x-a)^4 = \frac{15}{16 \cdot 24}(.1)^4 \approx 3.9 \times 10^{-6}$. (Note: sticking strictly to the method of the textbook, we find the maximum of $|f^{(4)}(x)| = \frac{15}{16}x^{-7/2}$ on the interval $[0.9, 1.1]$. In this method substitute $x = 0.9$ to find the bound M.)

18. Use Taylor's inequality to find a reasonable bound for the error in approximating the quantity $e^{0.60}$ with a third degree Taylor polynomial for e^x centered at $a = 0$. We are estimating e^x at $x = 0.6$. For $f(x) = e^x$, $n = 3$, $a = 0$, $x = 0.6$, Taylor's inequality gives the bound $\dfrac{Mx^4}{4!}$, where M is the maximum of $|f^4(x)| = |e^x|$ on the interval $(0, 0.6)$. Since $|f^4(x)| = e^x$ is an increasing function, its maximum on this interval occurs at the right-hand endpoint, so $M = e^{0.6}$. The bound is: $\dfrac{e^{0.6}(0.6)^4}{4!} < \dfrac{3^{0.6}(0.6)^4}{4!}$. (Note: sticking strictly to the method of the textbook, we would find the maximum of $|f^{(4)}x|$ on the interval $(-0.6, 0.6)$, and the same value of $M = e^{0.6}$ would work.)

19. Consider the error in using the approximation $\sin\theta \approx \theta - \theta^3/3!$ on the interval $[-1, 1]$. Where is the approximation an overestimate? Where is it an underestimate?

For $0 \le \theta \le 1$, the estimate is an underestimate (the alternating Taylor series for $\sin\theta$ is truncated after a negative term). For $-1 \le \theta \le 0$, the estimate is an overestimate (the alternating Taylor series is truncated after a positive term).

20. Write down from memory the Taylor Series centered around $a = 0$ for the functions e^x, $\sin x$, $\cos x$ and $\frac{1}{1-x}$.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, converges to e^x on $(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$, converges to $\sin x$ on $(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, converges to $\cos x$ on $(-\infty, \infty)$
$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, converges to $\frac{1}{1-x}$ on $(-1, 1)$

21. (a) Find the 4th degree Taylor Polynomial for $f(x) = \sqrt{x}$ centered at $a = 1$ by differentiating and using the general form of Taylor Polynomials.
$P_4(x) = 1 + \frac{x-1}{2} - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4$
(b) Use the previous answer to find the 4th degree T.P. for $f(x) = \sqrt{1-x}$ centered at $x = 0$.
substitute $1 - x$ for x, need 4th degree: $P_4(x) = 1 - \frac{x}{2} - \frac{1}{8}x^2 - \frac{1}{16}x^3 - \frac{5}{128}x^4$
(c) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x}}$.
Differentiate, multiply by -2: $P_3(x) = 1 + \frac{x}{2} + \frac{3}{8}x^2 + \frac{5}{16}x^3$
(d) Use the previous answer to find the 3rd degree T.P. for $f(x) = \frac{1}{\sqrt{1-x^2}}$.
Substitute x^2 for x: $P_3(x) = 1 + \frac{x^2}{2}$, note that the x^3 term is 0.
(e) Use the previous answer to find the 3rd degree T.P. for $f(x) = \arcsin x$.
Integrate, substitute to verify that the constant term is 0: $P_3(x) = x + \frac{x^3}{6}$