MATH 2300 – review problems for Exam 1

1. Evaluate the integral $\int \sin x \cos x \, dx$ in each of the following ways:

   (a) Integrate by parts, with $u = \sin x$ and $dv = \cos x \, dx$. The integral you get on the right should look much like the one you started with, so you can solve for this integral. (Some people call this the "boomerang" method.)

   (b) Integrate by parts, with $u = \cos x$ and $dv = \sin x \, dx$.

   (c) Substitute $w = \sin x$.

   (d) Substitute $w = \cos x$.

   (e) First use the fact that $\sin x \cos x = \frac{1}{2}\sin(2x)$, and then antidifferentiate directly.

   (f) Show that answers to parts (a)–(e) of this problem are all the same. It may help to use the identities $\cos^2 x + \sin^2 x = 1$ and $\cos(2x) = 1 - 2\sin^2 x$.

2. Let $f(x)$ be a continuous function on the set of all real numbers. Show that

$$\int_0^1 f(e^x)\, e^x \, dx = \int_1^e f(x)\, dx.$$

3. (a) Explain why the integral

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}}$$

   is improper.

   (b) Show that

$$\int_2^5 \frac{x\, dx}{\sqrt{x^2 - 4}} = \sqrt{21}.$$

4. Suppose that $\int_0^1 f(t)dt = 5$. Calculate the following:

   (a) $\int_0^{0.5} f(2t)dt$

   (b) $\int_0^1 f(1-t)dt$

   (c) $\int_1^{1.5} f(3-2t)dt$

5. Evaluate the following integrals:

   (a) $\displaystyle\int 2x\cos(x^2)\, dx$

   (b) $\displaystyle\int e^{2x}\sin(2x)\, dx$

   (c) $\displaystyle\int \cos^2\theta \, d\theta$

   (d) $\displaystyle\int x^2 \sin(x) dx$

   (e) $\displaystyle\int \frac{1}{x^2\sqrt{16 - x^2}}\, dx$

(f) $\displaystyle\int \frac{3x^2+6}{x^2(x^2+3)}\,dx$

(g) $\displaystyle\int \sqrt{25-x^2}\,dx$

(h) $\displaystyle\int \frac{3x-1}{x^2-5x+6}\,dx$

(i) $\displaystyle\int \sin^3(5x)\cos(5x)\,dx$

(j) $\displaystyle\int_2^3 \frac{x^2}{1+x^3}\,dx$

(k) $\displaystyle\int_0^3 x\,e^{x^2}\,dx$

(l) $\displaystyle\int x^7 e^{x^4}\,dx$

(m) $\displaystyle\int (\ln(x))^2\,dx$

6. Evaluate the following integrals

   (a) $\int y\sqrt{y^2+1}\,dy$

   (b) $\int y\sqrt{y+1}\,dy$

7. (a) Calculate $\displaystyle\int_2^4 \frac{dx}{(x-3)^2}$, if it exists.

   (b) Find $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{e^{2x}+1}\,dx$, if it converges.

8. Estimate $\displaystyle\int_1^2 \ln x\,dx$, by subdividing the interval $[1,2]$ into eight equal parts, and using:

   (a) A left-hand Riemann sum:

   (b) A right-hand Riemann sum:

   (c) The midpoint rule:

   (d) The trapezoid rule:

   (e) Simpson's rule:

   (f) Get a bound on the error for the above approximations using midpoint rule and trapezoid rule.

   (g) Evaluate the integral more exactly using technology, and comment on how your estimates compare to it.

9. Using the table, estimate the total distance traveled from time $t = 0$ to time $t = 6$ using the trapezoidal rule and the midpoint rule. Divide the interval $[0,6]$ into $n = 3$ equal parts. Next, if we know that $|f''(x)|$ is no bigger than 5 on the interval $[0,6]$, then find a bound for the error of your approximations.

Time, t	0	1	2	3	4	5	6
Velocity, v	4	5	6	8	9	5	3

10. Consider the function $f(x) = x^2 + 3$ on the interval $[0, 1]$. Determine whether each of the following four methods of integral approximation will give an overestimate or underestimate of $\int_0^1 f(x)dx$. In each case, draw a picture to justify your answer.

    (a) the left Riemann sum

    (b) the right Riemann sum

    (c) the trapezoidal rule

    (d) the midpoint rule

11. Suppose $f(x)$ is concave up and decreasing on the interval $[0, 1]$. Suppose the approximations LEFT(100), RIGHT(100), MID(100), and TRAP(100) yield the following estimates for $\int_0^1 f(x)\,dx$: 1.10, 1.25, 1.30, and 1.50, but *not necessarily in that order*. Which estimate do you think came from which method? Between which two estimates does the exact value of the integral lie? Please explain your reasoning.

12. Which of the following integrals can be integrated using partial fractions?

    (a) $\int \frac{1}{x^4-5x^2+4}\,dx$

    (b) $\int \frac{1}{x^4+1}\,dx$

    (c) $\int \frac{1}{x^3-8}\,dx$

    (d) $\int \frac{2x+1}{x^2+4}\,dx$

    Make sure you can show the partial fraction decompositions.

13. For this set of problems, state which techniques are useful in evaluating the integral. You may choose from: integration by parts; partial fractions; long division; completing the square; trig substitution; or another substitution. There may be multiple answers.

    (a) $\displaystyle\int \frac{x^2}{\sqrt{1-x^2}}\,dx$

    (b) $\displaystyle\int \frac{1}{\sqrt{6x-x^2-8}}\,dx$

    (c) $\displaystyle\int x\sin x\,dx$

    (d) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

    (e) $\displaystyle\int \frac{x^2}{1-x^2}\,dx$

    (f) $\displaystyle\int (1+x^2)^{-3/2}\,dx$

    (g) $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

    (h) $\displaystyle\int \frac{1}{1-x^2}\,dx$

14. A patient is given an injection of Imitrex, a migraine medicine, at a rate of $r(t) = 2te^{-2t}$ ml/sec, where t is the number of seconds since the injection started.

    (a) By using an improper integral, estimate the total quantity of Imitrex injected.

    (b) What fraction of this dose has the patient received at the end of 5 seconds?

15. Let f be a differentiable function. Suppose that $f''(0) = 1$, $f''(1) = 2$, $f'(0) = 3$, $f'(1) = 4$, $f(0) = 5$, $f(1) = 6$. Compute $\int_0^1 f(x)f'(x)dx$.

16. For some constants A and B, the rate of production $R(t)$ of oil in a new oil well is modelled by:

$$R(t) = A + Be^{-t}\sin(2\pi t),$$

    where t is the time in years, A is the equilibrium rate, and B is the“variable” coefficient.

    (a) Find the total amount of oil produced in the first N years of operation.

    (b) Find the average amount of oil produced per year over the first N years.

    (c) From your answer to part (b), find the average amount of oil produced per year as $N \to \infty$.

    (d) Looking at the function $R(t)$, explain how you might have predicted your answer to part (c) without doing any calculations.

    (e) Do you think it is reasonable to expect this model to hold indefinitely?

17. The rate, r, at which a population of bacteria grows can be modeled by $r = te^{3t}$, where t is time in days. Find the total increase in population of bacteria after 20 days.

18. Determine whether the following improper integrals converge or diverge.

    (a) $\displaystyle\int_1^{\infty} \frac{\cos^2 x}{\sqrt{x^3}}\,dx$

    (b) $\displaystyle\int_3^{\infty} \frac{1}{x^2 \ln x}\,dx$

    (c) $\displaystyle\int_1^{\infty} \frac{3+\sin x}{x^2}\,dx$

    (d) $\displaystyle\int_3^{\infty} \frac{5+2\sin x}{x-2}\,dx$

    (e) $\displaystyle\int_1^{\infty} \frac{\ln x}{x}\,dx$

    (f) $\displaystyle\int_1^{\infty} \frac{1}{x \ln x}\,dx$

    (g) $\displaystyle\int_{10}^{\infty} \frac{1}{x^2-9}\,dx$

19. The following integrals represent the area of some region in the xy plane. Draw a possible graph of the region, labeling the axes and giving the equation(s) of the function(s).

    (a) $\displaystyle\int_{-2}^{0} (-4x)\,dx$

    (b) $\displaystyle\int_{-3}^{3} (-\sqrt{9-x^2})\,dx$

    (c) $\displaystyle\int_1^{2} 3y\,dy$

    (d) $\displaystyle\int_0^{1} (\sqrt{y}-y)\,dy$

20. Using slices parallel to the base, write a definite integral representing the volume of a cone with a height of 10 cm and a base of diameter 6 cm.

21. Consider the region bounded by $y = \sqrt{x}$, $y = 0$, $x = 1$.

    (a) Sketch the region and find its area
    (b) Sketch the solid obtained by rotating the above region around the x-axis.
    (c) Write an integral that gives the volume of the solid
    (d) Repeat with the same region rotated around the y-axis.
    (e) Repeat with the same region rotated around the line $x = -5$.

(f) Repeat with the same region rotated around the line $y = 1$.

22. Find the volume of the solid whose base is the region in the xy-plane bounded by the curves $y = x$ and $y = x^2$ and whose cross sections perpendicular to the x-axis are squares with one side in the xy-plane.

23. Do the same thing as the previous problem except with semi-circle cross sections and then again with cross sections that are isosceles triangles of height 3. Notice that the integrals you get are just multiples of the integral from the previous problem. This means you don't have to evaluate them, instead you can just find the appropriate multiple of the answer to the previous problem!

24. Suppose $f(0) = 1$, $f(1) = e$, and $f'(x) = f(x)$ for all x. Find

$$\int_0^1 e^x f'(x)dx.$$