(4) a.) "Bacteria grows at a rate proportional to its size"

$$\frac{dP}{dt} = kP \qquad (k>0 \text{ because of growth})$$

b.) This is a seperable differentiel equation

$$\frac{1}{P}\,dP = k\,dt \qquad (\text{seperate "P"s and "t"s})$$

$$\int \frac{1}{P}\,dP = \int k\,dt \qquad (\text{integrate both sides})$$

$$\ln|P| = kt + c \qquad (\text{Solve for } P)$$

$$P = Ce^{kt} \qquad (\text{let } e^{c} = C)$$

c.) "Culture Contains 300 cells initially" $\Rightarrow P(0) = 300$ cells

"After 0.5 hour population increased to 540" $\Rightarrow P(0.5) = 540$ cells

Need to find C and k

$$P(0) = C = 300$$

letting $C = 300$

$$P(0.5) = 300e^{k/2} = 540$$

$$e^{k/2} = 9/5$$

$$k/2 = \ln(9/5)$$

$$k = 2\ln(9/5) \approx 1.176$$

Model for number of bacteria a time t is $P(t) = 300e^{1.176t}$

d.) $300e^{1.176t} = 10000$ (solve for t)

$$e^{1.176t} = 100/3$$

$$t = \frac{\ln(100/3)}{1.176} \approx 2.983 \text{ hours.}$$

5. a.) "decay at rate proportional to remaining mass"

$$\frac{dm}{dt} = km \quad ; \quad (k<0 \text{ because of decay})$$

b.) This is a seperable differential equation

$$\frac{1}{m}\,dm = k\,dt \qquad (\text{seperate "m"s and "t"s})$$

$$\int \frac{1}{m}\,dm = \int k\,dt \qquad (\text{integrate both sides})$$

$$\ln|m| = kt + c \qquad (\text{solve for } m)$$

$$m = Ce^{kt} \qquad (\text{Let } e^{c} = C)$$

Now we need to solve for C and k.

"We begin with a 50 mg sample" $\Rightarrow C = 50$

"Cobalt-60 has a ½-life of 5.24 years" $\Rightarrow 0.5 = e^{5.24k}$

if $0.5 = e^{5.24k}$

$$\frac{\ln(0.5)}{5.24} = k \quad \Rightarrow k \approx -0.132$$

Model for amount of Cobalt-60 after t years is $m(t) = 50e^{-0.132t}$

c.) After 20 years $m(20) = 50e^{-0.132\cdot 20} = 3.548$ mg

To find time it takes to decay to 1 mg

$$1 = 50e^{-0.132t} \qquad (\text{solve for } t)$$

$$e^{-0.132t} = 1/50$$

$$t = -\ln(1/50)/0.132 \approx 29.574 \text{ years.}$$

(6.) Newton's Law of Cooling

"Rate of Cooling proportional to temperature difference between object and its surroundings"

$$\frac{dT}{dt} = k(T - T_s)$$

$T(t)$ = temperature of object at time t.

T_s = temperature of Surroundings

Solve differential equation by Seperation of variables.

$$\frac{1}{T - T_s}\, dT = k\,dt \quad \text{(seperate } T \text{ and } t\text{)}$$

$$\int \frac{1}{T - T_s}\, dt = \int k\,dt \quad \text{(integrate both sides)}$$

$$\ln|T - T_s| = kt + c \quad \text{(solve for } T\text{)}$$

$$T(t) = Ce^{kt} + T_s$$

- If turkey starts at room temperature of 70° $\Rightarrow T(0) = 70°F$
- $T_s = 350°F$
- After 1 hr turkey is 100°F $\Rightarrow T(1) = 100°F$

Use this to solve for k and c

<u>To find C:</u> $70 = Ce^0 + 350 \Rightarrow C = -280$

<u>To find k:</u> $100 = -280e^k + 350$

$$\frac{25}{28} = e^k \Rightarrow k = \ln\left(\frac{25}{28}\right) \approx -0.113$$

Model for temperture of turkey at time t is $T(t) = -280e^{-0.113t} + 350$

Need to find time when turkey reaches 160°F.

$$160 = -280e^{-0.113t} + 350$$

$$\frac{19}{28} = e^{-0.113t} \Rightarrow t = \frac{-\ln(19/28)}{0.113} \approx 3.422 \text{ hours}$$

If turkey starts cooking at 11 am it will be done around 2:30 pm. Turn oven temp. down

7. Continuously Compounded interest modeled by

$$A(t) = A_0 e^{rt}$$

r - interest rate

A_0 - initial amount invested

- Find doubling time at 6% interest rate

$$2A_0 = A_0 e^{0.06t}$$ (solve for t)

$$2 = e^{0.06t}$$

$$\frac{\ln(2)}{0.06} = t$$ or $t \approx 11.553$ years

- Find doubling time at 3% interest rate

$$2A_0 = A_0 e^{0.03t}$$ (solve for t)

$$2 = e^{0.03t}$$

$$\frac{\ln(2)}{0.03} = t$$ or $t \approx 23.105$ years

(8.) Using the logistic differential equation we get

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

where

k - relative growth rate

M - carrying capacity

We are given $k = 0.002$ and $M = 15{,}000{,}000{,}000$.

The differential equation that models this situation is

$$\frac{dP}{dt} = 0.002\left(1 - \frac{P}{15{,}000{,}000{,}000}\right)$$

(9.) Let $P(t)$ = Percentage of CO_2 in Room

Initially room has 0.15% CO_2 so $P(0) = 0.15$.

rate-in $= (0.05)(2\ m^3/min) = 0.1\ m^3/min$

rate-out $= P(t)(2\ m^3/min) = 2P(t)\ m^3/min$

Thus

$$\frac{dP}{dt} = (\text{rate-in}) - (\text{rate-out}) = 0.1 - 2P$$

this is a seperable differential equation so

$$\frac{1}{0.1-2P}\,dP = dt \quad \Rightarrow -\ln|0.1-2P| = t + c$$

Now using $P(0) = 0.15$ we see $c = -\ln|0.1 - 0.3|$
$= -\ln(0.2)$

then we get

$$|0.1 - 2P| = 0.2e^{-t}$$

$P(t)$ is continuous $P(0) = 0.15$ and the right side is never zero implies $0.1 - 2P < 0$ so

$$2P - 0.1 = 0.2e^{-t}$$

$$\Rightarrow \quad P = 0.1e^{-t} + 0.05$$