CHAPTER 130

Review of complex analysis in one variable

This gives a brief review of some of the basic results in complex analysis. In particular, it outlines the background in single variable complex analysis that is discussed in [Huy05, §1.1].

1. Complex numbers

We define the complex numbers to be the field $(\mathbb{R}^2, +, \cdot)$ where $(\mathbb{R}^2, +)$ is the standard $\mathbb{R}$-vector space of dimension 2, and $\cdot$ is defined by $(a,b)\cdot(c,d) = (ac - bd, ad+bc)$. We will denote by $\mathrm{CO}(2,\mathbb{R})$ the group of real two by two conformal matrices:

$$\mathrm{CO}(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : (a,b) \in \mathbb{R}^2 - \{0\} \right\}.$$

Set

$$\widehat{\mathrm{CO}}(2,\mathbb{R}) = \mathrm{CO}(2,\mathbb{R}) \cup \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This is a ring under matrix addition and multiplication.

Exercise 130.1.1. *Show that there is an isomorphism of rings*

$$\phi : \mathbb{C} \to \widehat{\mathrm{CO}}(2,\mathbb{R})$$
$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Exercise 130.1.2. *Given a linear map $A : \mathbb{R}^2 \to \mathbb{R}^2$, there exists a linear map $\alpha \in M(1,\mathbb{C}) = \mathbb{C}$ making the following diagram commute*

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{A} & \mathbb{R}^2 \\ \| & & \| \\ \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C}. \end{array}$$

if and only if $A \in \widehat{\mathrm{CO}}(2,\mathbb{R})$. In this case $A = \phi(\alpha)$. In particular, given $a+ib \in \mathbb{C}$, then multiplication of complex numbers by $a+ib$, when viewed as an $\mathbb{R}$-linear map of $\mathbb{R}^2$, is given by $\phi(a+ib)$.

2. Holomorphic maps

Definition 130.2.3 (Holomorphic map)**.** *Let $U \subseteq \mathbb{C}$ be an open subset. A map*

$$f : U \to \mathbb{C}$$

is said to be holomorphic if it at each point $p \in U$, the real differential $D_p f : \mathbb{R}^2 \to \mathbb{R}^2$ exists and is complex linear (i.e., $D_p f \in \widehat{\mathrm{CO}}(2,\mathbb{R})$).

EXAMPLE 130.2.4. A complex analytic function on an open subset of the complex plane is holomorhpic (on that open subset). We will recall below the proof that the converse holds.

EXAMPLE 130.2.5. In particular, the function $e^z := \sum_{n=0}^{\infty} z^n$ is analytic on $\mathbb{C}$, and is therefore holomorphic.

Corollary 130.2.6. *Let $U \subseteq \mathbb{C} = \mathbb{R}^2$ be an open subset. A map $f : U \to \mathbb{C}$ that is differentiable at each point $p \in U$ is holomorphic if and only if, writing $f(x,y) = u(x,y) + iv(x,y)$, the Cauchy–Riemann equations*

$$\frac{\partial u}{\partial x}(p) = \frac{\partial v}{\partial y}(p), \quad \frac{\partial u}{\partial y}(p) = -\frac{\partial v}{\partial x}(p)$$

hold at each point $p \in U$.

PROOF. This follows immediately from the definitions. $\square$

REMARK 130.2.7. The Cauchy–Riemann equations imply that if we define $\frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$, then a differentiable function $f(z)$ is holomorphic on an open set U if and only if $\frac{\partial}{\partial \bar{z}} f(z) = 0$ for every $z \in U$.

Recall that if Γ is a (positively oriented) smooth contour in the complex plane, parameterized by a smooth map $\gamma : [a,b] \to \mathbb{C}$, then

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

EXAMPLE 130.2.8. The main example is:

$$\int_{B_\epsilon(0)} z^n dz = \begin{cases} 2\pi i & n = -1 \\ 0 & n \neq -1. \end{cases}$$

The main bound that one uses repeatedly is (e.g., [Rud87, §10.8 Eq. (5), p.202]):

$$\left|\int_\Gamma f(z)dz\right| \leq \int_\Gamma |f(z)|ds \leq \sup_{z\in\Gamma} |f(z)||\Gamma| \tag{84}$$

where $|\Gamma|$ is the length of the path Γ.

Lemma 130.2.9. *Let $U \subseteq \mathbb{C}$ be an open subset. A function $f : U \to \mathbb{C}$ is holomorphic if and only if for every $B_\epsilon(z_0) \subseteq U$ we have*

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0}dz. \tag{85}$$

PROOF. We sketch the proof. Suppose first that f is holomorphic. The key point is that the function $f(z)/(z - z_0)$ is holomorphic everywhere in $B_\epsilon(z_0)$ except for the point z_0. Therefore, using say Stoke's Theorem, the integral in (85) is the same for every positively oriented circle $C_r(z_0) := \partial B_r(z_0)$ of positive radius contained in the disk $B_\epsilon(z_0)$. Now consider:

$$\int_{C_r} \frac{f(z)}{(z-z_0)}dz = \int_{C_r} \frac{f(z_0)}{(z-z_0)} + \int_{C_r} \frac{f(z) - f(z_0)}{(z-z_0)} = 2\pi i f(z_0) + \int_{C_r} \frac{f(z) - f(z_0)}{(z-z_0)}dz.$$

Using the bound on the modulus of the integral (84), and taking the limit as r goes to 0, one obtains (85).

Conversely, suppose that (85) holds for every disk. Then expanding the geometric series in the integrand, and passing the integral through the uniformly convergent series, we see that f is analytic. As pointed out above, analytic functions are holomorphic. $\square$

Corollary 130.2.10. *Let $U \subseteq \mathbb{C}$ be an open subset. Then $f : U \to \mathbb{C}$ is holomorphic if and only if it is complex analytic.*

PROOF. This is contained in the proof above. $\square$

REMARK 130.2.11. The proof above shows that if $f : U \to \mathbb{C}$ is just assumed to be continuous, then

$$f(z_0) = \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\partial B_\epsilon(z_0)} \frac{f(z)}{z - z_0} dz.$$

2.1. Basic properties of holomorphic maps. Here we review a few basic facts about holomorphic maps.

2.1.1. *Local structure theorem.*

Theorem 130.2.12 (Local structure theorem)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map. Then locally, f factors as $z \mapsto z^m$, followed by a holomorphic isomorphism.*

REMARK 130.2.13. More precisely we mean that for each point $p \in U$, there is a ball $B_\epsilon(p)$, so that $f|_{B_\epsilon(p)} : B_\epsilon(p) \to f(B_\epsilon(p))$ factors as

$$B_\epsilon(p) \xrightarrow{z \mapsto (z-p)^m} B_{\epsilon^m}(0) \xrightarrow{g} f(B_\epsilon(p))$$

where g is a holomorphic isomorphism.

PROOF. To make the formulas simpler, take $p = 0$. We have immediately from analyticity that $f(z) = z^m h(z)$, where $h(z)$ is nowhere vanishing in a neighborhood of 0. We claim there is $g(z)$ such that $g(z)^m = h(z)$. In short, using that $h(z)$ is nowhere vanishing, and possibly taking a smaller neighborhood, we can define a branch of log and set $g(z) = \exp(\frac{1}{m} \log h(z))$. $\square$

REMARK 130.2.14. To avoid technicalities with logs, just observe that $h'(z)/h(z)$ is holomorphic near 0. Therefore, using analyticity of holomorphic functions, we can find $a(z)$ such that $a'(z) = h'(z)/h(z)$. Then we have $\frac{d}{dz}(h(z)e^{-a(z)}) = 0$, so that $h(z) = e^{a(z)}$. Then we set $g(z) = e^{a(z)/m}$.

REMARK 130.2.15. The number m is determined uniquely at a point $p \in U$ by the number of preimages of f near $f(p)$, or equivalently, by the order of vanishing of f at p.

Corollary 130.2.16. *The zero set of a nonconstant holomorphic function has no limit points in the domain of definition.*

PROOF. This follows immediately from the structure theorem (and the elementary case of $z \mapsto z^m$). $\square$

2.1.2. *Open mapping theorem.*

Theorem 130.2.17 (Open mapping)**.** *Nonconstant holomorphic maps are open (take open sets to open sets).*

PROOF. This follows immediately from the local structure theorem. □

2.1.3. *Maximum principle.*

Theorem 130.2.18 (Maximum principle)**.** *Let $U \subseteq \mathbb{C}$ be open and connected. If $f : U \to \mathbb{C}$ is holomorphic and non-constant, then $|f|$ has no local maximum in U. If U is bounded and f can be extended to a continuous function $f : \overline{U} \to \mathbb{C}$, then $|f|$ takes its maximal values on the boundary ∂U.*

PROOF. Use the open mapping theorem. □

2.1.4. *Identity theorem.*

Theorem 130.2.19 (Identity theorem)**.** *If $f, g : U \to \mathbb{C}$ are two holomorphic functions on a connected open subset $U \subseteq \mathbb{C}$ such that $f(z) = g(z)$ for all z in a non-empty open subset $V \subseteq \mathbb{C}$, then $f = g$.*

REMARK 130.2.20. There are stronger versions of the identity theorem (e.g., take any subset V with limit points), but in this form it immediately generalizes to higher dimensions.

PROOF. From the corollary to the local structure theorem we have that zero sets of nonconstant holomorphic functions have no limit points (in the domain of definition). To prove the identity theorem, take the difference of the two functions and consider the zero set. □

2.1.5. *Riemann extension theorem.*

Theorem 130.2.21 (Riemann extension theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a bounded holomorphic function on a punctured disk. Then f can be extended to a holomorphic function $f : B_\epsilon(z_0) \to \mathbb{C}$.*

PROOF. The boundedness shows that $g(z) = (z - z_0)^2 f(z)$ is complex differentiable at z_0, and therefore given by a power series. Since $g(z)$ vanishes to at least order 2 at z_0, we have that $f(z)$ is analytic. □

2.1.6. *Riemann mapping theorem.*

Theorem 130.2.22 (Riemann mapping theorem)**.** *Let $U \subseteq \mathbb{C}$ be a simply connected proper open subset. Then U is biholomorphic to the unit ball $B_1(0)$; i.e., there exists a bijective holomorphic map $f : U \to B_1(0)$ such that its inverse f^{-1} is also holomorphic.*

PROOF. We refer the reader to [**Rud87**, Thm. 14.8, p.283]. □

2.1.7. *Liouville's theorem.*

Theorem 130.2.23 (Liouville's Theorem)**.** *Every bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ is constant. In particular, there is no biholomorphic map between $\mathbb{C}$ and a ball $B_\epsilon(0)$ with $\epsilon < \infty$.*

PROOF. Using analyticity, it is not hard to show that

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Then on a circle C_R of radius R centered about z_0, if $|f(z)| \le M_R$ for all $z \in C_R$, then the derivatives of f at z_0 satisfy (for each $n \in \mathbb{N}$)

$$\left| f^{(n)}(z_0) \right| \le \frac{n! M_R}{R^n}.$$

Apply this to the first derivative, and let $R \to \infty$. □

2.1.8. *Residue theorem.*

Theorem 130.2.24 (Residue theorem)**.** *Let $f : B_\epsilon(z_0)^* \to \mathbb{C}$ be a holomorphic function on a punctured disk. Then f can be expanded in a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and the coefficient a_{-1} is given by the residue formula*

$$a_{-1} = \frac{1}{2\pi i} \int_{\partial B_{\epsilon/2}(z_0)} f(z) dz.$$

PROOF. The existence of the Laurent series follows from basic results on analytic functions (on annuli, we can take the sum of two analytic functions, where for one, we invert z). Then we just integrate and use uniform convergence. □

2.1.9. *Inverse function theorem.*

Theorem 130.2.25 (Inverse function)**.** *Let $f : U \to \mathbb{C}$ be a holomorphic map. If $f'(z_0) \neq 0$, then f is locally a holomorphic isomorphism near z_0.*

PROOF. Use the real inverse function theorem, and the fact that the inverse of a conformal matrix is conformal. □

REMARK 130.2.26. Using the structure theorem, one can show that if a holomorphic function is injective, then it is biholomorphic onto its image (indeed, locally it must be $z \mapsto z$, so it is locally biholomorphic; but it is a bijection so it is globally biholomorhpic). The same result will hold for maps $f : U \subseteq \mathbb{C}^n \to \mathbb{C}^n$, but may fail if the dimensions of the source and target are not the same; e.g., $z \mapsto (z^3, z^2)$ is holomorphic and injective, but not biholomorphic onto its image.

2.1.10. *Schwarz lemma.*

Lemma 130.2.27 (Schwarz Lemma)**.** *Let f be a holomorphic function on an open neighbourhood of the closure of the disk $B\epsilon(0)$. Assume that f vanishes to order k at the origin. If there is some real number C such that $|f(z)| \le C$ for all $z \in \overline{B_\epsilon(0)}$, then actually there is the possibly stronger bound:*

$$|f(z)| \le C \left(\frac{|z|}{\epsilon} \right)^k$$

for all $z \in \overline{B_\epsilon(0)}$.

REMARK 130.2.28. In short, we know the maximum, say C, of $|f(z)|$ occurs on the boundary circle $C_\epsilon(0)$. However, we can estimate how much smaller the modulus of f is on the interior, by multiplying C by the fraction of the distance we are to the boundary circle (to the power k).

PROOF. Fix $z \in B_\epsilon(0)$ and define a holomorphic function $g_z(w)$ as follows: For $w \leq \epsilon$ one sets

$$g_z(w) := w^{-k} f\left(w \cdot \frac{z}{|z|}\right).$$

Then $|g_z(w)| \leq \epsilon^{-k} C$ for $|w| = \epsilon$. The maximum principle then implies the same bound $g_z(w)| \leq \epsilon^{-k} C$ for all $|w| \leq \epsilon$. Hence

$$|z|^{-k}|f(z)| = |g_z(|z|)| \leq \epsilon^{-k} C,$$

giving the desired bound. □

3. Meromorphic functions

Let $U \subseteq \mathbb{C}$ be open. A meromorphic function f on U is an equivalence class of functions on the complement of a nowhere dense (i.e., closure has empty interior) subset $S \subseteq U$ (e.g., S has no limit points in U) with the following property: There exist:

- an open cover $U = \bigcup_{i \in I} U_i$,
- holomorphic functions $g_i, h_i : U_i \to \mathbb{C}$,

satisfying

$$h_i|_{U_i - S} \cdot f|_{U_i - S} = g_i|_{U_i - S}$$

for every i.

More precisely: we consider pairs (S, f) where S and $f : U - S \to \mathbb{C}$ are as above. We say that $(S, f) \sim (S', f')$ if setting $S'' = S \cup S'$, then with $h_i|_{U_i - S''} \cdot f'|_{U_i - S''} = g_i|_{U_i - S''}$.

REMARK 130.3.29. The point is that S is nowhere dense if and only if $\overline{U - \overline{S}} = U$.

$$\begin{aligned}
\overline{U - \overline{S}} \subsetneq U &\iff \exists\, x \in U \text{ with } x \notin \overline{U - \overline{S}} \\
&\iff \exists\, x \in U \text{ and } V \subseteq U \text{ open nbhd of } x \text{ with } V \cap (U - \overline{S}) = \emptyset \\
&\iff \exists\, x \in U \text{ and } V \subseteq U \text{ open nbhd of } x \text{ with } V \subseteq \overline{S} \\
&\iff \exists V \subseteq U \text{ open with } V \subseteq \overline{S}.
\end{aligned}$$

REMARK 130.3.30. If S has no limit points then it is nowhere dense. If S has no limit points, then it is closed. If S had nonempty interior, then it would have limit points. Thus if S has no limit points, it is nowhere dense. On the other hand, in $B_1(0)$, take $S = \{1/n : n \geq 2\}$. This set has closure with empty interior, but it has a limit point, 0.