3.1. Determining the Jordan form. It is easy to determine the Jordan form of a matrix, provided one knows the eigenvalues. One can also use this to establish the uniqueness of the Jordan form, up to permuting the blocks.

Theorem 9.3.8 (Computing Jordan forms)**.** *Let $A \in \mathrm{M}_n(K)$, let $\lambda_1, \dots, \lambda_m \in \overline{K}$ ($m \le n$) be the distinct eigenvalues of A, and assume that all the eigenvalues of A lie in K. For $i = 1, \dots, m$, let*

$$T_{\lambda_i} := A - \lambda_i I,$$
$$d_{\lambda_i}^{(j)} := \dim \ker T_{\lambda_i}^j.$$

Then we have that

$$d_{\lambda_i}^{(j)} = \sum_{\ell=1}^{j} \#\{\text{Jordan blocks for } \lambda_i \text{ of size at least } \ell\}.$$

PROOF. We leave the proof of the theorem as an exercise. □

REMARK 9.3.9. From the theorem we have,

$$\begin{aligned} d_{\lambda_i}^{(1)} &= \#\{\text{Jordan blocks for } \lambda_i\} \\ d_{\lambda_i}^{(2)} - d_{\lambda_i}^{(1)} &= \#\{\text{Jordan blocks for } \lambda_i \text{ of size at least } 2\} \\ &\vdots \\ d_{\lambda_i}^{(j)} - d_{\lambda_i}^{(j-1)} &= \#\{\text{Jordan blocks for } \lambda_i \text{ of size at least } j\}. \end{aligned}$$

Moreover, if $d_{\lambda_i}^{(j)} = d_{\lambda_i}^{(j+1)}$, then $d_{\lambda_i}^{(j)} = d_{\lambda_i}^{(j+k)}$ for all $k \ge 1$.

REMARK 9.3.10. Knowing the multiplicities of the eigenvalues as roots of the characteristic polynomial, one can reduce the amount of computation needed to find the Jordan form. For instance, if λ_i has multiplicity m_i as a root of the characteristic polynomial, then there can be at most m_i Jordan blocks for λ_i, with equality holding if and only if each Jordan block is size 1. Similarly, the maximum size of a Jordan block for λ_i is m_i, with equality holding if and only if there is only one Jordan block for λ_i.

Example 9.3.11. Let us compute a Jordan form for the matrix

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

using the algorithm above; there are faster methods, but this will illustrate the general algorithm. First, we find the characteristic polynomial

$$p_A(t) = (1-t)(2-t)(2-t),$$

so that the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 2$. We then start computing the $d_{\lambda_i}^{(j)}$.

$$d_{\lambda_1}^{(1)} = \dim \ker(A - I) = \dim \ker \begin{bmatrix} 0 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \dim \ker \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1.$$

The matrix computation was to reduce to row echelon form, from which it is easy to compute the dimension of the kernel. Similarly, one can compute

$$d_{\lambda_1}^{(2)} = \dim\ker(A-I)^2 = 1,\ \ d_{\lambda_1}^{(3)} = \dim\ker(A-I)^3 = 1.$$

Thus there is one Jordan block for the eigenvalue 1, and this Jordan block is of size 1.

Next we compute:

$$d_{\lambda_2}^{(1)} = \dim\ker(A-2I) = \dim\ker\begin{bmatrix} -1 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 1.$$

$$d_{\lambda_2}^{(2)} = \dim\ker(A-2I)^2 = 2.$$

$$d_{\lambda_2}^{(3)} = \dim\ker(A-2I)^3 = 2.$$

Thus we conclude that there is 1 Jordan block for the eigenvalue 2, and this Jordan block is size 2. In short, we have that a Jordan form for A is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

Of course, we could have deduced this immediately from the fact that 2 is a root of multiplicity 2 for the characteristic polynomial, and that $\dim\ker(A-2I) = 1$.