Exercise 14.1.21 ([Art11, Ch. 6, Ex. 7.8]). *Decompose the set* $\mathrm{M}_2(\mathbb{C})$ *of* 2×2 *matrices into orbits for the following operations of* $\mathrm{GL}_2(\mathbb{C})$*:*

(1) *Left multiplication;*
(2) *Conjugation.*

Solution. (1) For left multiplication, the orbits are indexed by matrices in reduced row echelon form. That is the orbits are

- $\left\{ A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : A \in \mathrm{GL}_2(\mathbb{C}) \right\} = \mathrm{GL}_2(\mathbb{C})$.
- For each $x \in \mathbb{C}$ there is an orbit

$$\left\{ A \begin{bmatrix} 1 & x \\ 0 & 0 \end{bmatrix} : A \in \mathrm{GL}_2(\mathbb{C}) \right\} = \left\{ \begin{bmatrix} a_{11} & xa_{11} \\ a_{21} & xa_{21} \end{bmatrix} : (a_{11}, a_{21}) \neq (0,0) \in \mathbb{C}^2 \right\}.$$

- $\left\{ A \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} : A \in \mathrm{GL}_2(\mathbb{C}) \right\} = \left\{ \begin{bmatrix} 0 & a_{11} \\ 0 & a_{21} \end{bmatrix} : (a_{11}, a_{21}) \neq (0,0) \in \mathbb{C}^2 \right\}$.
- $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

The reason for this is that the elementary matrices generate $\mathrm{GL}_2(\mathbb{C})$, and multiplication on the left by elementary matrices corresponds to elementary row operations. Since every matrix can be put into reduced row echelon form with elementary row operations, and this reduced row echelon form is unique, there is a unique reduced row echelon matrix in each orbit.

(2) For conjugation, the orbits are given by the Jordan form matrices (up to equivalence, of permuting the Jordan form). That is, the orbits are

- For each subset $\{\lambda_1, \lambda_2\} \subseteq \mathbb{C}$ (i.e., for each nonempty subset of order 1 or 2) there is an orbit
$$\left\{ A \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} A^{-1} : A \in \mathrm{GL}_2(\mathbb{C}) \right\}.$$
Note that the orbit depends only on the set $\{\lambda_1, \lambda_2\}$, and not on the ordering of the elements.
- For each $\lambda \in \mathbb{C}$ there is an orbit
$$\left\{ A \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix} A^{-1} : A \in \mathrm{GL}_2(\mathbb{C}) \right\}.$$

This follows immediately from the theorem on Jordan forms for matrices.

Exercise 14.1.22 ([Art11, Ch. 6, Ex. 10.2]). *Let* S *be a finite set on which a group* G *acts transitively. Let* $U \subseteq S$ *be a subset. Show that if* $x, y \in S$*, then the order of the set* $\{gU : g \in G,\ x \in gU\}$ *is the same as the order of the set* $\{gU : g \in G,\ y \in gU\}$*.*

Solution. First observe that $\{gU : g \in G,\ x \in gU\}$ is in fact finite, since it is a subset of the finite set $\mathscr{P}(S)$, of subsets of S. Next let us define a bijection

$$\phi : \{gU : g \in G,\ x \in gU\} \longrightarrow \{gU : g \in G,\ y \in gU\};$$

once we establish the existence of such a bijection, we will be done.

Since G acts transitively on S, by definition, there is an element $h \in G$ such that $y = gh$. We can therefore define the map ϕ by setting

$$\phi(gU) := hgU,$$

since if $x \in gU$, then $y \in hgU$. To see that ϕ is injective, we can define an inverse map ψ by the rule

$$\psi(gU) = h^{-1}gU,$$

since $x = h^{-1}y$, so that if $y \in gU$, then $x \in h^{-1}gU$. These maps are inverses, since

$$\psi\phi(gU) = h^{-1}hgU = gU, \quad \text{and} \quad \phi\psi(gU) = hh^{-1}gU = gU.$$