SAMPLE
MIDTERM I
INTRODUCTION TO LINEAR ALGEBRA

MATH 3130

Friday February 15, 2013

Name

Please answer all of the questions, and show your work.
All solutions must be explained clearly to receive credit.

1	2	3	4	5	6	7	8	
10	10	20	10	20	10	10	10	Total

Date: February 9, 2013.

1
10 points

1. Let $(V, +, \cdot)$ be a vector space. Using only the definition of a vector space, show that $0 \cdot v = \mathscr{O}$ for all $v \in V$, where $\mathscr{O}$ is the additive identity of V.

2
10 points

2. Consider the system of linear equations

$$\begin{array}{rcrcrcr} 3x_1 & + & 9x_2 & + & 27x_3 & = & -3 \\ -3x_1 & - & 11x_2 & - & 35x_3 & = & 5 \\ 2x_1 & + & 8x_2 & + & 26x_3 & = & -4 \end{array}$$

This system of linear equations can be written in the form

$$Ax = b$$

where A is a 3×3 matrix, and x and b are both 3×1 matrices. What are the matrices A, x and b?

3
20 points

3. Using the matrices A and b from the previous problem, put the augmented matrix

$$[A|b]$$

in reduced row echelon form using only elementary row operations. (You must show each of your steps. Use the back of this page if necessary).

4
10 points

4. Let A be the matrix from Problem 2.

4.(a). (3 points) Are the columns of A linearly independent? Explain.

4.(b). (3 points) Are the rows of A linearly independent? Explain.

4.(c). (2 points) What is the column rank of A? Explain.

4.(d). (2 points) What is the row rank of A? Explain.

5
20 points

5. Let A be the matrix from Problem 2. The matrix A defines a linear map

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

given by the rule $f(x) = Ax$.

5.(a). (5 points) Find the kernel of f.

5.(b). (2 points) What is the dimension of the kernel of f?

5.(c). (2 points) What is the dimension of the image of f?

5.(d). (1 point) Is f an isomorphism?

6
10 points

6.(a). Verify $(2, -1, 0)$ is a solution to the system of linear equations in Problem 2.

6.(b). Find all solutions to the system of linear equations in Problem 2.

7
10 points

7. Let T be the set of functions $f : \mathbb{R} \to \mathbb{R}$ such that f'' exists. It is a fact that T forms a sub-vector space of the vector space $\mathrm{Map}(\mathbb{R}, \mathbb{R})$ consisting of all functions $f : \mathbb{R} \to \mathbb{R}$.

7.(a). Show that the map

$$F : T \to \mathrm{Map}(\mathbb{R}, \mathbb{R})$$

given by the rule

$$F(f) = f''$$

for all $f \in T$ is a linear map.

7.(b). What is the kernel of F?

8
10 points

8. True or False. Mark **T** for true and **F** for false.

☐ **8.(a)** Let $A \in M_{m\times n}(\mathbb{R})$. If the columns of A span $\mathbb{R}^m$, then for any $b \in \mathbb{R}^m$ there is an $x \in \mathbb{R}^n$ such that $Ax = b$.

☐ **8.(b)** Let $A \in M_{m\times n}(\mathbb{R})$. If the columns of A are linearly independent, then for any $b \in \mathbb{R}^m$ there is at most one $x \in \mathbb{R}^n$ such that $Ax = b$.

☐ **8.(c)** Let $A \in M_{m\times n}(\mathbb{R})$. There is an $x \in \mathbb{R}^n$ such that $Ax = 0$.

☐ **8.(d)** If a linear map is injective, then it is an isomorphism.

☐ **8.(e)** Let $f : V \to V'$ be a linear map of vector spaces. Suppose there exists a linear map $g : V' \to V$ such that $g \circ f = Id_V$. Then f is an isomorphism.

☐ **8.(f)** The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ for all $x \in \mathbb{R}$ is a linear map.

☐ **8.(g)** If $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then $A^n = \begin{bmatrix} 1 & 2^{n-1} \\ 0 & 1 \end{bmatrix}$.

☐ **8.(h)** If A and B are $m \times n$ matrices, then $A + B = B + A$.

☐ **8.(i)** If A and B are $m \times m$ matrices, then $AB = BA$.

☐ **8.(j)** Every vector space is isomorphic to $\mathbb{R}^n$ for some non-negative integer n.