SAMPLE FINAL EXAM
INTRODUCTION TO LINEAR ALGEBRA

MATH 3130

May, 2013

Name	

Please answer all of the questions, and show your work.
All solutions must be explained clearly to receive credit.

Date: April 30, 2013.

1
10 points

1.(a). Linear map. Give the definition of a linear map from a vector space $(V, +, \cdot)$ to a vector space $(V', +', \cdot')$.

1.(b). Kernel. Give the definition of the kernel of a linear map.

1.(c). Show that the kernel of a linear map is a sub-vector space of the source.

2
10 points

2. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 5x_2 & + & 4x_3 & + & x_4 & = & -3 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 5 \\ x_1 & - & 4x_2 & + & 6x_3 & + & 2x_4 & = & 10 \end{array}$$

This system of linear equations can be written in the form

$$Ax = b$$

where A is a 3×4 matrix, x is a 4×1 matrix and b is a 3×1 matrix. What are the matrices A, x and b?

3
10 points

3. Let A and b be the matrices in the previous problem. Put the augmented matrix

$$[A|b]$$

in reduced row echelon form.

4
10 points

4. Find all solutions to the system of equations

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 5x_2 & + & 4x_3 & + & x_4 & = & -3 \\ x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 5 \\ x_1 & - & 4x_2 & + & 6x_3 & + & 2x_4 & = & 10 \end{array}$$

5
10 points

5. Let $V \subseteq \mathrm{Map}(\mathbb{R}, \mathbb{R})$ be the sub-vector space of continuous functions. Let W be the sub-vector space of V spanned by the functions 1, $\sin x$, and $\sin^2 x$.

5.(a). Show that $1, \sin x, \sin x^2 \in W$ are linearly independent.

5.(b). What is the dimension of W?

5.(c). Consider the map $L : W \to W$ defined by the rule

$$L(a_1 \cdot 1 + a_2 \cdot \sin x + a_3 \cdot \sin^2 x) = (a_1 + a_2) \cdot 1 + (a_1 - a_2) \cdot \sin x + (a_3 - a_1) \cdot \sin^2 x$$

for all $a_1, a_2, a_3 \in \mathbb{R}$. For instance $L(1) = 1 + \sin x - \sin^2 x$. Show that L is a linear map.

5.(d). Write down the matrix for L in terms of the ordered basis $(1, \sin x, \sin^2 x)$.

5.(e). Is L an isomorphism?

5.(f). Show that $1, \sin x, \cos 2x \in V$ also form a basis of W. [Hint: Use an identity from trigonometry. If you can not remember the identity, then deduce it from $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\sin^2\theta + \cos^2\theta = 1$.]

5.(g). Write down the change of coordinates matrix from the ordered basis $(1, \sin x, \sin^2 x)$ into the ordered basis $(1, \sin x, \cos 2x)$.

5.(h). Let A be the matrix for L in terms of the ordered basis $(1, \sin x, \sin^2 x)$. Let B be the matrix for L in terms of the ordered basis $(1, \sin x, \cos 2x)$. Find a matrix S so that $B = S^{-1}AS$. (You do not need to find B.)

6
10 points

6. Consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{pmatrix}$$

6.(a). What is the characteristic polynomial of A?

6.(b). What are the eigenvalues of A?

6.(c). For each eigenvalue λ in part (a), find a basis for the eigenspace associated to λ.

6.(d). If A is diagonalizable, find a diagonal matrix D and an invertible matrix S such that $A = S^{-1}DS$. If A is not diagonalizable, explain why not.

7
10 points

7. Suppose that $v \in \mathbb{R}^n$. Show that if $v.w = 0$ for all $w \in \mathbb{R}^n$, then $v = 0$.

8
10 points

8. Let A be a the matrix.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -1 & 2 \end{pmatrix}$$

8.(a). What is the dimension of the kernel of A?

8.(b). What is the dimension of the image of A?

8.(c). Find a basis for the image of A.

8.(d). Find a 3×3 matrix P so that for any $v \in \mathbb{R}^3$, the orthogonal projection of v onto the image of A is given by Pv.

9
10 points

9. Let

$$P = \begin{pmatrix} .1 & .5 \\ .9 & .5 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} .2 \\ .8 \end{pmatrix}$$

Find

$$\lim_{n\to\infty} P^n v.$$

10
10 points

10. True or false.

☐ **10.(a)** The row space of a matrix is the same as the row space of the reduced row echelon form of the matrix.

☐ **10.(b)** A change of coordinates matrix is invertible.

☐ **10.(c)** Suppose that P is an $n \times n$ matrix with non-negative entries, and such that the column sums are equal to 1. Then the same is true for P^2.

☐ **10.(d)** If v and w are eigenvectors for an $n \times n$ matrix A, both with eigenvalue λ, then $v + w$ is an eigenvector for A with eigenvalue λ.

☐ **10.(e)** An $n \times n$ matrix has n linearly independent eigenvectors if and only if it has n distinct eigenvalues.

☐ **10.(f)** The characteristic polynomial of an $n \times n$ matrix has degree n.

☐ **10.(g)** Every matrix can be diagonalized.

☐ **10.(h)** Let $v, w \in \mathbb{R}^n$. If θ is the angle between v and w, then $\cos\theta = \dfrac{v.w}{||v||||w||}$.

☐ **10.(i)** Let W be a sub-vector space of $\mathbb{R}^n$. The orthogonal complement of W is the set

$$W^\perp = \{v \in \mathbb{R}^n : w.v = 0 \text{ for some } w \in W\}.$$

☐ **10.(j)** Let $A \in M_{m\times n}(\mathbb{R})$. Then $\ker A^T A = \ker A$.