Analytic Geometry and Calculus 2
MATH 2300
Wednesday April 11, 2012
Sample Midterm 3 (Solutions)

On my honor as a University of Colorado at Boulder student I have neither given nor received unauthorized assistance on this exam.

Name:____

DO NOT OPEN THIS EXAM UNTIL INSTRUCTED TO DO SO!

- ◯ **001** Martinez .... (8am)
- ◯ **002** Spina .... (9am)
- ◯ **003** Rosenbaum .... (10am)
- ◯ **004** Shannon .... (11am)
- ◯ **005** Casalaina-Martin .... (11am)
- ◯ **006** Scherer .... (12pm)
- ◯ **007** Davison .... (1pm)
- ◯ **008** Wayne .... (1pm)

You may NOT use: books, notes, or calculators.

You SHOULD use: complete sentences and clear handwriting.

In order to receive full credit your answer must be **complete**, **legible** and **correct**. Show all of your work, and give clear explanations.

DO NOT WRITE IN THIS BOX!

Problem	Points	Score
1	20 pts	
2	20 pts	
3	20 pts	
4	20 pts	
5	20 pts	
TOTAL	100 pts	

1
20 points

1. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent.

1.(a). $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

SOLUTION: This series is CONDITIONALLY CONVERGENT. Since $\lim_{n\to\infty} 1/n = 0$, the series converges by the **alternating series** test. On the other hand, since the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges, this series is not absolutely convergent.

1.(b). $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n}$

SOLUTION: This series is ABSOLUTELY CONVERGENT. This is a geometric series $\sum_{n=0}^{\infty} x^n$ with $x = 1/3$, and so converges (to $3/2$). Since all the terms are positive, it is absolutely convergent.

1.(c). $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{2^n}{n^2+1}$

SOLUTION: This series is DIVERGENT. $\lim_{n\to\infty}\frac{2^n}{n^2+1} \neq 0$ (use say l'Hospital's rule).

1.(d). $\sum_{n=0}^{\infty}\frac{n!}{n^n}$

SOLUTION: This series is ABSOLUTELY CONVERGENT. Applying the ratio test we consider

$$\lim_{n\to\infty}\frac{\left|\frac{(n+1)!}{(n+1)^{n+1}}\right|}{\left|\frac{n!}{n^n}\right|} = \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\frac{n^n}{n!} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \left(\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right)^{-1} = \frac{1}{e} < 1.$$

Thus the series converges. All the terms are positive, and so the series is absolutely convergent.

2
20 points

2. Find the **radius of convergence** and the **interval of convergence** for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$

[Hint: remember to check the endpoints.]

SOLUTION: **The radius of convergence is** 1 **and the interval of convergence is** $[1,3)$.

To see this, we can use the Ratio Test. We consider

$$\lim_{n\to\infty} \frac{\left|\frac{(x-2)^{n+1}}{n+1}\right|}{\left|\frac{(x-2)^n}{n}\right|} = \lim_{n\to\infty} \frac{|x-2|^{n+1}}{n+1}\frac{n}{|x-2|^n} = \lim_{n\to\infty} |x-2|\frac{n}{n+1} = |x-2| < 1$$

if and only if $1 < x < 3$. Thus the radius of convergence is 1. For the left endpoint, $x = 1$, the series becomes an alternating series $\sum_{n=0}^{\infty}(-1)^n/n$, which is convergent. On the other hand, for the right endpoint, $x = 3$, the series becomes the harmonic series, which does not converge.

Alternatively, one could first show (as above) that the series converges for $x = 1$ and diverges for $x = 3$. From this it follows that the radius of convergence is 1, and the endpoints are already determined.

3
20 points

3. Consider the function $f(x) = \dfrac{2}{(1-x)^3}$.

3.(a). Find the degree 2 Taylor polynomial for $f(x)$ centered at 0. [Hint: $f(x) = 2(1-x)^{-3}$.]

SOLUTION: The degree 2 Taylor polynomial for $f(x)$ centered at 0 is:

$$2 + 6x + 12x^2$$

(See the next page)

3.(b). Find the Taylor series for $f(x)$ centered at 0.

SOLUTION: The Taylor series for $f(x)$ centered at 0 is given by

$$\sum_{n=0}^{\infty}(n+2)(n+1)x^n.$$

Indeed, $f(x) = 2(1-x)^{-3}$, $f'(x) = 6(1-x)^{-4}$, $f''(x) = 4!(1-x)^{-5}$, and

$$f^{(n)}(x) = \frac{(n+2)!}{(1-x)^{n+3}}.$$

This gives $f^{(n)}(0) = (n+2)!$, from which one concludes

$$\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty}\frac{(n+2)!}{n!}x^n = \sum_{n=0}^{\infty}(n+2)(n+1)x^n.$$

Alternatively, setting $g(x) = \frac{1}{1-x}$, we can observe that $f(x) = g''(x)$. Since $g(x) = \sum_{n=0}^{\infty} x^n$, we have

$$f(x) = \frac{d}{dx}\frac{d}{dx}\sum_{n=0}^{\infty}x^n = \frac{d}{dx}\sum_{n=1}^{\infty}nx^{n-1} = \sum_{n=2}^{\infty}n(n-1)x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)x^n.$$

4
20 points

4. Consider the function $f(x) = e^{2x}$.

4.(a). [10 points] Find the degree 3 Taylor polynomial for $f(x) = e^{2x}$ centered at 0.

SOLUTION:

$$1 + 2x + 2x^2 + \frac{4}{3}x^3$$

Recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}$, so that

$$f(x) = e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!}.$$

4.(b). [2 points] Use the fact that $e^{1/5} < 3/2$ to show that

$$\frac{(2^4 e^{1/5})\left(\frac{1}{10}\right)^4}{4!} < 10^{-4}.$$

[This will be useful in (c).]

SOLUTION:

$$\frac{(2^4 e^{1/5})\left(\frac{1}{10}\right)^4}{4!} = \frac{16e^{1/5}}{24}10^{-4} < \frac{16}{24}\cdot\frac{3}{2}10^{-4} = 10^{-4}.$$

(Although it is not required to check $e^{1/5} < 3/2$, one can do this by noting that $e^{1/5} < 3^{1/5} < 3/2$; the left inequality follows from $e < 3$ and the right from $32 = 2^5 < 81 = 3^4$.)

4.(c). [8 points] Let $P_3(x)$ be the degree 3 Taylor polynomial for $f(x) = e^{2x}$ centered at 0. Use (b) to show that for all $-\frac{1}{10} \le x \le \frac{1}{10}$,

$$|f(x) - P_3(x)| < 10^{-4}.$$

[Hint: the left hand side of the inequality in (b) may show up in a standard error estimation.]

SOLUTION: We start by observing that $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$, $f''(x) = 4e^{2x}$ and

$$f^{(n)}(x) = 2^n e^{2x}.$$

These are increasing, positive functions of x. Thus on the interval $[-1/10, 1/10]$ we have

$$|f^{(4)}(x)| \le |f^{(4)}(1/10)| = 2^4 e^{1/5}.$$

Using the Lagrange error bound on the interval $[-1/10, 1/10]$, we have

$$|f(x) - P_3(x)| \le \frac{(2^4 e^{1/5})|x|^4}{4!} \le \frac{(2^4 e^{1/5})\left(\frac{1}{10}\right)^4}{4!}.$$

This is less than 10^{-4} by the previous problem.

5
20 points

5. Answer the following problems on differential equations.

5.(a). Is $y = \sin 2x$ a solution to the differential equation $y'' - 4y = 0$? Explain your answer.

SOLUTION: No, $y = \sin 2x$ is not a solution to the differential equation. We have $y'' = -4\sin 2x$, and $y'' - 4y = -8\sin 2x \neq 0$.

5.(b). Find a solution to the differential equation $\dfrac{dy}{dx} = \dfrac{x^2+1}{y}$, with $y(0) = 1$.

SOLUTION: $y = \sqrt{\frac{2}{3}x^3 + 2x + 1}$ is a solution to the differential equation, with $y(0) = 1$. Separating the variables, we have "$ydy = (x^2+1)dx$" so that

$$\int y dy = \int (x^2+1)dx.$$

This gives

$$\frac{1}{2}y^2 = \frac{1}{3}x^3 + x + C.$$

Consequently,

$$y = \sqrt{\frac{2}{3}x^3 + 2x + C}.$$

Setting $y(0) = 1$, we see that $C = 1$. Thus

$$y = \sqrt{\frac{2}{3}x^3 + 2x + 1}$$

is a solution with $y(0) = 1$.