ABSTRACT ALGEBRA 1 (3140) CLASS NOTES

SEBASTIAN CASALAINA-MARTIN

1. Lecture 1

1.1. **Preliminaries.** This is Abstract Algebra 1, MATH 3140. My name is Sebastian Casalaina-Martin; you may call me Yano, which is short for Sebastiano.

All information for the class is available online at my webpage:

http://math.colorado.edu/∼sbc21/

(This will come up as one of the first hits if you type "casalaina", or "yano math boulder" into google).

My email is

`casa@math.colorado.edu`

Please check the website regularly for homework assignments. Also, there is a syllabus, as well as a webpage with class policies. Please read these both, and let me know within the first two weeks of class if you have any special conditions of which I need to be aware, or any conflicts with scheduled exams due to religious observances.

The textbook for the course is ***A First Course in Abstract Algebra, Seventh Edition*** **by J.B. Fraleigh** [2].

1.2. **Historical discussion.** I want to begin today's lecture by giving a (quasi)-historical introduction to the field of algebra, motivating the material we will cover in this class. This history is taken from N. Bourbaki, *Elements of mathematics, Algebra I, Chapters 1-3*, pp. 180-190 [1]. I refer you all there for more detailed references to the literature.

Among the most fundamental mathematical notions is that of composition (that is addition, multiplication, etc.). Documents from the ancient Egyptians and Babylonians show that they already had a complete system of rules for making computations with (positive) natural numbers, (positive) rational numbers, lengths and areas. It is worth noting that in the documents that have survived, there is no indication of concern for justifying these rules; their validity appears to have been held from the empirical fact that the rules led to the correct solutions.

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This changed with the ancient Greeks, and their axiomatic approach to mathematics (particularly geometry). For instance, in Euclid's *Elements*, he gives formal proofs of a number of rules of calculation. In contrast, and somewhat surprisingly, the Greek's focus on mathematical rigor led to some regression in the techniques of computation (one of the main problems was that all numbers were associated with lengths and areas, and hence made abstract computation involving relations of degree greater than two more difficult).

It was only with the decline of the classical Greek mathematicians that abstract computation returned as a field of study for the Greeks. Professional calculators, called "logisticians", had continued during the classical era to apply (without formal justification) the rules inherited from the Egyptians and Babylonians. Recall that the Egyptians had been "civilized" for around two thousand years before the rise of the Classical Greek civilization, and the Greeks had borrowed many things from their (initially at least) more advanced neighbors. In Greece, Diophantus, discarding the notion that numbers must represent geometric quantities, developed rules of abstract computation; one of these, in modern language, is equivalent to the formula $x^m x^n = x^{m+n}$. It also appears that Diophantus was the first to use a letter to represent an unknown in an equation, and begun calculations with negative numbers.

One of the first steps forward after this was to develop better mathematical notation. The invention of zero and the negative numbers in India during the High Middle Ages, and the creation of the imaginary number by the Italian algebraists of the 16th century, are important examples. One of the striking features of these "inventions" was that initially, they were introduced formally, without clear understanding of their meaning outside of their definitions. They were often called "false", "absurd", "impossible", etc.

I now quote the following passage from Bourbaki:

> For the Greeks of the Classical Period, enamoured above all of clear thought, such extensions were inconceivable; they could only arise with [people] more disposed than were the Greeks to display a somewhat mystic faith in the power of their methods and to allow themselves to be carried along by the mechanics of their calculations without investigating whether each step was well founded; a confidence moreover usually justified *a posteriori*, by the exact results to which the extension led.

Although the Indian mathematicians were already aware of interpretations of negative numbers in certain cases (such as commercial debt), in the following centuries, as the methods and results of the Greek and Indian mathematicians spread to the West via the Arabs, the manipulation of these numbers became more familiar, and took on further concrete interpretations (for instance in terms motion).

Meanwhile, in the ninth century, progress in algebra was made in the middle east. The word "algebra" in fact comes from the Arabic word "al-jabr". After a hiatus in Europe, the "Dark Ages", interest in Algebra developed again with the progess of the Italian school in the 16th century. In their work, they gave a solution "by radicals" of equations of the 3rd and 4th degrees. It was at this point that they felt compelled, despite their aversions, to introduce the imaginary numbers. A little later, in the seventeenth century, Descartes and Viete "perfected" algebraic notation, which is more or less the notation we use today.

From the middle of the 17th century to the end of the 18th century, mathematicians focused primarily on the development of calculus. This began to change in the 19th century, with the work of Lagrange, Gauss, Cauchy and others. One of the problems which had been confounding mathematicians at the time was that despite the passage of two centuries, no solution "by radicals" had been found to equations of degrees five and higher. The reason for this was settled by an 18 year old French mathematician named Evariste Galois.

Many of the biographies written about Galois are considered to be apocryphal. The following account is taken from J. Gallian *Contemporary Abstract Algebra*, pp.168-169 [3], and I make no claim to the validity of what follows.

Galois was born in 1811, near Paris. He took his first course in mathematics at age 15, and then quickly mastered the work of Legendre and Lagrange. At 18, Galois wrote his seminal work on the theory of equations, to which I alluded a moment ago. This work introduced what is now called Galois Theory, a central subject in mathematics, which has led to a tremendous amount of further research. Many of the basic notions in Group Theory and Field Theory that we will study in this class trace their roots back to Galois.

Galois is believed to have had a somewhat colorful personality, and life story. He twice failed the entrance exam to l'Ecole Polytechnique. He apparently did not know some basic mathematics, and did almost all computations in his head, making it difficult for the examiners to follow what he was doing. It is said that during one exam, he became so annoyed at the stupidity of the examiner, that he threw an eraser at him (something hopefully I will not have to experience from any of you). Galois spent most of the last year and a half of his as a political prisoner during the French Revolution. In 1832, Galois was shot in a duel, and died the next day at the age of 20.

One of the main goals of this class will be to study the foundations of abstract algebra in order to understand Galois' proof that there is no solution "in radicals" to polynomials of degree five or greater.

1.3. **Review of sets.** See Fraleigh [2].

1.4. **Introduction to groups.** See Fraleigh [2].

References

1. N. Bourbaki, *Algebra I. Chapters 1–3*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 1998, Translated from the French, Reprint of the 1989 English translation [MR0979982 (90d:00002)].
2. J.B. Fraleigh, *A First Course in Abstract Algebra, Seventh Edition*, Addison Wesley, Boston, 2003.
3. J.A. Gallian, *Contemporary Abstract Algebra, Third Edition*, D.C. Heath, Toronto, 1994.

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