MATH 232: HOMEWORK 4

DUE FRIDAY MARCH 21

Problems

(1) Show that every irreducible surface $S \subset \mathbb{P}^n$ (possibly singular) with $\deg(S) \leq n-2$ lies in a hyperplane.

(2) Let $S \subset \mathbb{P}^n$ be an irreducible surface (possibly singular) of degree $n-1$, not lying in a hyperplane. Show that S is one of the following:

  (a) A cone over a projectively normal curve of degree $n-1$ in $\mathbb{P}^{n-1}$.

  (b) The Veronese surface.

  (c) The surface $\mathbb{F}_r$ embedded by $|h+kf|$ where $r = n-1-2k$, $k \geq 1$.

  [Hint: if S is singular, it is a cone by the previous problem. If it is smooth, show that a smooth hyperplane section H of S is rational, then that the linear system $|K_S + 2H|$ is base point free. Deduce that either $K_S^2 = 9$ and we are in case (b), or $K_S^2 = 8$ and we are in case (c).]

(3) Let S be a (smooth) surface for which $-K_S$ is ample. Show that either $S = \mathbb{P}^1 \times \mathbb{P}^1$ or S is obtained from $\mathbb{P}^2$ by blowing up r distinct points, $r \leq 8$ in general position. [Hint: Show that S is rational using Castelnuovo's theorem, and use the fact that if S dominates $\mathbb{F}_n$ for $n \geq 2$, then $-K_S$ is not ample.]

(4) Let $S \subset \mathbb{P}^n$ be a surface with hyperplane section $H \sim -K_S$. Show that S is either a del Pezzo surface S_d ($3 \leq d \leq 9$), or the surface S_8', the image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^8$ by the linear system $|2h_1 + 2h_2|$, where the h_i are the classes of the rulings on $\mathbb{P}^1 \times \mathbb{P}^1$.