MATH 232: HOMEWORK 3

DUE WEDNESDAY MARCH 12

Problems

(1) Let C be a curve, E a rank two vector bundle on C, and set $S = \mathbb{P}E$, with projection $\pi : S \to C$.

  (a) Let $s \in S$, and set $F = \pi^{-1}(s)$. Show that on the blow-up of S at s, the strict transform of F can be contracted, and that the surface obtained is a geometrically ruled surface S'. The rational map $S \dashrightarrow S'$ is called an elementary transformation.

  (b) Give another proof, as follows, of the fact that a minimal ruled surface which is not rational, is isomorphic to a geometrically ruled surface: Fix X to be a minimal surface, which is not rational, with a birational map

$$\phi : X \dashrightarrow S.$$

  Show that ϕ is composed of an isomorphism and some elementary transformations. [Hint: if $n(\phi)$ is the minimum number of blow-ups necessary to make ϕ everywhere defined, show that there exists an elementary transformation $t : PE \dashrightarrow PE'$ such that $n(t \circ \phi) < n(\phi)$.]

(2) Set E_n to be the vector bundle on $\mathbb{P}^1$ with sheaf of sections $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-n)$, and set $\mathbb{F}_n = \mathbb{P}E_n$. Let $\mathscr{O}_{\mathbb{F}_n}(-1)$ be the tautological bundle, and let $s : \mathbb{P}^1 \to \mathbb{F}_n$ be the section induced from the natural inclusion $\mathscr{O} \to E_n$. Set $B = s(\mathbb{P}^1)$.

  (a) Show that the elementary transformation of $\mathbb{F}_n$ with center $s \in \mathbb{F}_n$ is isomorphic to either $\mathbb{F}_{n-1}$ or $\mathbb{F}_{n+1}$ depending on the position of s.

  (b) Show that the linear system $|\mathscr{O}_{\mathbb{F}_n}(1)|$ induces a morphism $f : \mathbb{F}_n \to \mathbb{P}^{n+1}$ which is an embedding outside of B, and contracts B to a point p. Show moreover, that $f(\mathbb{F}_n)$ is the cone with vertex p over the rational normal curve of degree n.

(3) Suppose that $f : X \to Y$ is a morphism of projective varieties, whose generic fiber is connected.

(a) If Y is smooth, show that every fiber of f is connected. [Hint: use the Stein factorization, and the fact that if $\phi : A \to B$ is a finite morphism of varieties, with B smooth (or more generally normal) then $\#\phi^{-1}(b) \leq \deg(\phi)$ for all $b \in B$.]

(b) Give an example of a morphism $f : X \to Y$ of projective varieties, whose generic fiber is connected, but with a fiber which is not connected.