MATH 232: HOMEWORK 2

DUE WEDNESDAY MARCH 5

Problems

(1) Let $X = X' = \mathbb{P}^2$, with homogeneous coordinates X_0, X_1 and X_2, and consider the Cremona transformation
$$X \dashrightarrow X'$$
given by
$$(X_0, X_1, X_2) \mapsto (X_1X_2, X_0X_2, X_0X_1).$$
 (a) Let $\pi : \hat{X} \to X$ be the blow-up of X at the points $p_1 = (0, 0, 1)$, $p_2 = (0, 1, 0)$, and $p_3 = (1, 0, 0)$, with exceptional divisors E_i respectively. Show that the induced rational map $\hat{X} \dashrightarrow X'$ extends to give a morphism.
 (b) Let $L_i = \{X_i = 0\}$, and set $\hat{L}_i$ to be the strict transform of L_i in $\hat{X}$. Show that $\hat{X} \to X'$ is the blow up of X' at three points, and that the L_i are the three exceptional divisors for that map.

(2) Let $X \subset \mathbb{P}^3$ be a nonsingular surface of degree 4, and $C \subset X$ an irreducible curve. Prove that if $C^2 < 0$, then $C^2 = -2$.

(3) Let X be a nonsingular curve and D the diagonal in $X \times X$. Show that $D^2 = -\deg K_X$.

(4) For every $k \in \mathbb{Z}$, show that there exists a smooth surface X, and a smooth curve $C \subset X$, with $C^2 = k$. Show also that there exists a surface Y, such that for all $k \in \mathbb{Z}$ there is a divisor D on Y with $D^2 = k$. [Hint: construct X and Y by blowing up points in $\mathbb{P}^2$.]

(5) Suppose that $f : \mathbb{P}^2 \to \mathbb{P}^1$ is given by homogeneous forms P and Q of degree d. How many blow-ups does one have to perform to resolve this rational map?

(6) Let C be an irreducible curve on a smooth surface S, $p \in C$, and $\hat{C}$ the strict transform of C in $\hat{S}$, the blow-up of S at p. The proximate points of p on C (or infinitely near points of order 1) are the points of $\hat{C}$ lying over p; their multiplicity is by definition their multiplicity as points of $\hat{C}$. The infinitely near

points to p of order n are the proximate points of the infinitely near points of order $(n-1)$.

(a) Show that
$$\operatorname{mult}_p(C) = \hat{C}.E = \sum m_x(\hat{C} \cap E) \geq \operatorname{mult}_x(\hat{C}),$$
where the sum runs through the proximate points $x \in \hat{C} \cap E$. Find an example with strict inequality.

(b) If C and C' are distinct irreducible curves, show that
$$m_p(C \cap C') = \sum_x \operatorname{mult}_x(C).\operatorname{mult}_x(C')$$
where x runs over the infinitely near points of p on C and C'.

(c) Let C denote the normalization of C. Show that
$$p_a(C) = g(N) + \sum_i \frac{1}{2} m_i(m_i - 1)$$
where m_i denotes the multiplicities of the points of C, including the infinitely near points.

(7) Suppose that S is a smooth surface, and $\pi : \hat{S} \to S$ is the blow-up of S at a point p, with exceptional divisor E. Prove that if
$$f : \hat{S} \to X$$
is a morphism, and $f(E)$ is a point, then the induced rational map
$$\phi = f \circ \pi^{-1} : S \dashrightarrow X$$
is a morphism.