MATH 232: FINAL EXAM

DUE MAY 14

Questions

All varieties will be assumed to be smooth projective varieties over the complex numbers, unless otherwise indicated. Please turn your exam in to me, in my office (SC 334), or email me a pdf copy.

(1) Let $f : X \to B$ be a surjective morphism of varieties (not necessarily projective) such that $X_b := f^{-1}(b)$ is a smooth projective minimal surface for all $b \in B$. If there exists $b_0 \in B$ such that X_{b_0} is an Enriques surface, show that X_b is an Enriques surface for all $b \in B$.

(2) Let E be a rank two vector bundle on a curve C, such that $H^0(C, E^\vee) \neq 0$, and $H^0(C, E^\vee \otimes L) = 0$ for all $L \in \mathrm{Pic}^d(C)$ with $d < 0$. Let $X = \mathbb{P}E$, with map $\pi : X \to C$. Let $S \to \pi^* E$ be the universal sub-bundle on X. Let F be a fiber of π.

   (a) In terms of the basis $\{\xi = [S^\vee], f = [F]\}$ for $N_1(X)_{\mathbb{R}}$, find $NE(X)$ and $\overline{NE}(X)$.

   (b) In terms of the basis $\{\xi = [S^\vee], f = [F]\}$ for $N^1(X)_{\mathbb{R}}$, find $Amp(X)$ and $Nef(X)$.

   (c) Find the K_X-negative extremal rays of $\overline{NE}(X)$, and the contraction morphisms associated to them.

   [Hint: Consider the cases $\deg(E) \geq 0$, and $\deg(E) < 0$ separately.]

(3) Let X be a surface with $-K_X$ very ample. For each such surface, describe $\overline{NE}(X)$; in particular, find the number of K_X-negative extremal rays, and the associated contractions.

(4) Show that any surface with an infinite number of exceptional curves is rational. Show that such a surface exists.

   [Hint: Let X be the blow-up of $\mathbb{P}^2$ at the nine points of intersection of two general plane cubics. Let E_1 and E_2 be the exceptional curves lying over two of these points of intersection. Show that the automorphism of X induced by $E_1 - E_2$ does not have finite order.]

(5) Let $\sigma : \mathbb{A}^n \to \mathbb{A}^n$ be given by $(x_1, \ldots, x_n) \mapsto (\zeta x_1, \ldots, \zeta x_n)$, for some primitive d-th root of unity ζ. Then σ induces an

action of $\mathbb{Z}_d$ on $\mathbb{A}^n$; let $X = \mathbb{A}^n/\mathbb{Z}_d$. Show that X has terminal singularities if $n > d$, and canonical singularities if $n \geq d$.

(6) Let $X \subset \mathbb{P}^3$ be quintic surface with a unique singular point $p \in X$, which is an ordinary double point (locally analytically near p, X looks like $V(x_1^2 + x_2^2 + x_3^2)$). Let X' be the blow-up of X at p. Find the minimal and canonical models of X'.