Homework 8

Due Friday, December 8

Exercises

1. Let H_g be the set of isomorphism classes of triples

$$(X, H, (\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g)),$$

where $X = V/\Lambda$ is a complex torus, $H \in NS(X)$ is a positive definite Hermitian form on V, and $(\lambda_1, \dots, \lambda_g, \mu_1, \dots, \mu_g)$ is a symplectic basis of Λ with respect to ImH. Let $\mathcal{H}_g$ be the Siegel upper half space:

$$\mathcal{H}_g = \{M \in M_{g\times g}(\mathbb{C}) : M = M^t,\ ImM > 0\}.$$

In class we showed there was a bijection $\mathcal{H}_g \to H_g$. For $Z \in \mathcal{H}_g$ we will denote the image of Z in H_g by (X_Z, H_Z, S_Z). There is an action of the symplectic group $Sp_{2g}(\mathbb{Z})$ on $\mathcal{H}_g$ given as follows: for

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in Sp_{2g}(\mathbb{Z}),$$

define $A \cdot Z = (\alpha Z + \beta)(\gamma Z + \delta)^{-1}$. For $Z, Z' \in \mathcal{H}_g$, show that $(X_Z, H_Z) \cong (X_{Z'}, H_{Z'})$ if and only if there exists $A \in Sp_{2g}(\mathbb{Z})$ such that $A \cdot Z = Z'$.

2. Given smooth curves C and C' of genus g, show that C_{g-1} is birational to C'_{g-1} if and only if $C \cong C'$.

3. Let k be an algebraicly closed field of characteristic p, and let $d > 0$ be an integer not divisible by p. Denote by $\mu_d \subset k$ the subgroup of d-th roots of unity, and consider the action of μ_d on $k[x_1, \dots, x_n]$ given as follows: for $\zeta \in \mu_d$ and $f \in k[x_1, \dots, x_n]$ set

$$\zeta \cdot f(x_1, \dots, x_n) = f(\zeta^{-1}x_1, \dots, \zeta^{-1}x_n).$$

This induces an action of μ_d on $\mathbb{A}^n_k$, and set $X = \mathbb{A}^n_k/\mu_d$. Show that there is a unique singular point $x \in X$, and that the projective tangent cone to X at x, $\mathbb{P}C_x X$, is the d-uple embedding of $\mathbb{P}^{n-1}$. Conclude that $mult_x X = d^{n-1}$. In particular, if $k = \mathbb{C}$ and $d = 2$, $mult_x X = 2^{n-1}$.

4. Let $N^g_k \subseteq \mathcal{A}_g$ be the locus of ppav's of dimension g whose theta divisor has a singular locus of dimension at least k. Let $\bar{J}_g \subseteq \mathcal{A}_g$ be the closure of the Jacobian locus. A result of Beauville's states that $N^4_0 \neq \bar{J}_4$. Use this to show that for all $g \geq 4$, $N^g_{g-4} \neq \bar{J}_g$.