Homework 5

Due Friday, November 3

Exercises

1. Let $f : X \to Y$ be a morphism of smooth varieties, and let V be a codimension one subvariety of Y. Show that
$$mult_x f^* V \geq mult_{f(x)} V$$
for all $x \in X$. Show that equality holds if and only if $f_* T_x X \not\subseteq C_{f(x)} V$, the tangent cone to V at $f(x)$.

For the following problems we will look at the difference map ϕ_d: let C be a smooth curve of genus g. Define
$$\phi_d : C_d \times C_d \to Pic^0(C)$$
by $\phi_d(D, E) = \mathcal{O}_C(D - E)$. The image will be denoted by V_d.

2. Show that the differential of ϕ_d has maximal rank $\min(2d, g)$ at a general point, and hence that $\dim(V_d) = \min(2d, g)$.

3. Show that
$$\phi_1 : C \times C \to V_1$$
is birational if C is not hyperelliptic, and has degree two if C is hyperelliptic. More generally, show that if C is not hyperelliptic, then ϕ_d is birational for $d < g/2$.

4. For C non-hyperelliptic, and $d \leq 2g$, find the class of V_d.

5. Suppose that C is a non-hyperelliptic curve of genus four, and suppose that $L \in Pic^0(C)$. Show that
$$\dim \phi_2^{-1}(L) = \begin{cases} 2 & L \cong \mathcal{O}_C \\ 1 & L \in V_1 \cup (K - 2W_3^1(C)) \\ 0 & otherwise \end{cases}$$
(the middle equation means that $L \cong \mathcal{O}_C(p - q)$ or else $L \cong M \otimes N^{-1}$ where M and N are line bundles of degree three, with $h^0 = 2$.