Homework 4

Due Friday, October 27

Exercises

1. Let C be a smooth curve, Σ_d be the symmetric group on d letters, and let $C_d = C^d/\Sigma_d$ be the symmetric product; i.e. the quotient of the cartesian product by the action of the symmetric group. Show that C_d is smooth, and $(\mathbb{P}^1)_d \cong \mathbb{P}^d$.

2. For a smooth curve C, a finite set of points $p_i \in C$, and a corresponding set of complex numbers n_i, show there is a meromorphic 1-form ω on C with a simple pole at each p_i, and no other poles, and with residue n_i at p_i, if and only if $\sum_i n_i = 0$.

3. Let (A, Θ) be a ppav of dimension g. Since $\Theta \in H^2(A, \mathbb{Z})$, we have that
$$\frac{\Theta^d}{d!} \in H^{2d}(A, \mathbb{Q})$$
for all positive integers $1 \leq d \leq g$. Show that in fact $\frac{\Theta^d}{d!} \in H^{2d}(A, \mathbb{Z})$. Show also that $\frac{\Theta^d}{d!}$ is minimal, i.e. there does not exist a class $\alpha \in H^{2d}(A, \mathbb{Z})$ and a positive integer a such that $a\alpha = \frac{\Theta^d}{d!}$.

4. Let C be a smooth curve of genus g, with Jacobian (JC, Θ). Show that if $g = 2$, then $\Theta \cong C$. If $g = 3$, then $\Theta \cong C_2$ unless C is hyperelliptic, in which case there is a unique singular point of multipliciy two, and C_2 is the blowup of Θ at the singular point. If $g = 6$ and there exists a point $x \in \Theta$ such that $mul_x\Theta = 3$, then either C is a plane quintic, or C is hyperelliptic.