Homework 1

Due Friday, October 6

Exercises

1. Let X be a complex torus of dimension g.

   (a) Compute the Hodge diamond of X.

   (b) Show that $NS(X)$ can have rank at most g^2.

2. Suppose $X = V/\Lambda$ is a complex torus of dimension g, and $x_1, \dots, x_{2g}$ are real coordinate functions with respect to an $\mathbb{R}$-basis $e_1, \dots, e_{2g}$ of V.

   (a) Show that the following diagram commutes

$$\begin{array}{ccc} H^2(\Lambda, \mathbb{C}) & \xrightarrow{\alpha} & Alt^2_{\mathbb{R}}(V, \mathbb{C}) \\ {\scriptstyle \phi_2}\big\downarrow & & {\scriptstyle \gamma_2}\big\downarrow \\ H^2(X, \mathbb{C}) & \xrightarrow{\rho} & H^2_{DR}(X), \end{array}$$

   where α is the extension of the isomorphism

$$\alpha : H^2(\Lambda, \mathbb{Z}) \to Alt^2(\Lambda, \mathbb{Z})$$

   given in class, ϕ_2 is the canonical isomorphism given in class, ρ is the de Rham isomorphism, and γ_2 is the canonical isomorphism which sends the alternating form E to the two form

$$-\sum_{1 \le i < j \le g} E(e_i, e_j) dx_i \wedge dx_j.$$

   (b) Conclude that the isomorphism $\alpha\phi_2^{-1} : H^2(X, \mathbb{Z}) \to Alt^2(\Lambda, \mathbb{Z})$ coincides with the isomorphism given by the cup product.

(c) Let H be a hermitian form on V, and let χ be a semicharacter for H. We have seen in class that

$$\alpha\phi_2^{-1}c_1(L(H,\chi)) = ImH \in Alt^2(\Lambda,\mathbb{Z}).$$

Let $\iota : H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C})$ be the natural inclusion. Conclude from part (a) that

$$\rho\iota c_1(L(H,\chi)) = -\sum_{1\le i<j\le g} ImH(e_i,e_j)dx_i \wedge dx_j.$$

(d) Suppose that $v_1,\dots,v_g$ form a basis for V as a complex vector space, and let $z_1,\dots,z_g$ be complex coordinate functions with respect to this basis. Show that

$$\rho\iota c_1(L(H,\chi)) = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{g} H(v_i,v_j)dz_i \wedge d\overline{z}_j.$$

Thus $L(H,\chi)$ is a positive line bundle if and only if H is positive definite.

3. Suppose that E is an integral, skew-symmetric, bilinear form on $\Lambda \cong \mathbb{Z}^{2g}$. Show that there exists a basis $\lambda_1,\dots,\lambda_{2g}$ for Λ such that E has the form

$$E = \left(\begin{array}{c|c} 0 & \Delta \\ \hline -\Delta & 0 \end{array}\right)$$

where Δ is the diagonal matrix

$$\Delta = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_g \end{pmatrix},$$

where d_i are non-negative integers, and $d_i | d_{i+1}$ for all $1 \le i \le g-1$. We will say that E is of type $(d_1,\dots,d_g)$.

[Hint: See page 304 of Griffiths and Harris, or use the "elementary divisors theorem".]

4. Recall that a line bundle L on a complex torus $X = V/\Lambda$ of dimension g is said to be of type $(d_1,\dots,d_g)$ if $c_1(L) \in Alt^2(\Lambda,\mathbb{Z})$ is of this type.

In the notation of Exercise 5, show that if L is of type $(d_1, \ldots, d_g)$, then there is a symplectic basis of Γ, $\lambda_1, \ldots, \lambda_g, \mu_1, \ldots \mu_g$, with corresponding real coordinates on V, $x_1, \ldots, x_g, y_1, \ldots, y_g$, such that

$$\rho\iota c_1(L) = -\sum_{i=1}^{g} d_i dx_i \wedge dy_i.$$

5. Let $X = \mathbb{C}/\Lambda$, where $\Lambda = \langle 1, \tau \rangle$, the subgroup generated by 1 and τ, with $Im\tau > 0$. Let z be the standard coordinate on $\mathbb{C}$. For which values of τ is

$$\omega = \frac{i}{2} dz \wedge d\overline{z}$$

the first chern class of some line bundle on X.