Glossary

This glossary does not pretend to be a course in algebraic geometry. Its purpose is to put in one easily available place some of the notions and facts that are used in the text. It can also be used to test of your background: if you can read this glossary, even if all the assertions are not familiar, you should have enough background in algebraic geometry to read the text.

1. Schemes and fibered products

A **scheme** is a ringed space $(X, \mathcal{O}_X)$ that is locally of the form $\mathrm{Spec}(A)$, for A a commutative ring with unit. For most purposes in this book, one can restrict attention to the case where A has nice properties, such as Noetherian, or finitely generated over the ground ring or field.[5] Since fibered products play a prominent role, however, one cannot stay in a category of reduced or irreducible varieties.

An **open subscheme** of a scheme X is an open subset U of X with its structure sheaf $\mathcal{O}_U = \mathcal{O}_X\,|_U$. A **closed subscheme** Y of X is the support of a quasi-coherent sheaf $\mathcal{I}$ of ideals, with the structure sheaf $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{I}$. A **subscheme** Y of X is given by a locally closed subspace Y of X, which is a closed subscheme of the open subscheme $U = X \smallsetminus (\overline{Y} \smallsetminus Y)$.

Given schemes X, Y, and Z, with morphisms $f : X \to Z$ and $g : Y \to Z$, there is a **fibered product**, which is a scheme $X \times_Z Y$, together with two projections $p : X \times_Z Y \to X$ and $q : X \times_Z Y \to Y$, with the $f \circ p = g \circ q$. The fibered product is determined by the following universal property: for any scheme S and morphisms $u : S \to X$ and $v : S \to Y$ such that $f \circ u = g \circ v$, there is a unique morphism $(u, v) : S \to X \times_Z Y$ such that $u = p \circ (u, v)$ and $v = q \circ (u, v)$. If $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, and $Z = \mathrm{Spec}(C)$, then $X \times_Z Y = \mathrm{Spec}(A \otimes_C B)$. In general, $X \times_Z Y$ is constructed by patching (see below). For clarity, if other morphisms from X or Y to Z are in use, the notation $X\ {}_f{\times}_{Z,g}\, Y$ or $X\ {}_f{\times}_g\, Y$ may be used for this fibered product.

A diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

[5] We give the definitions in their natural generality, following [**EGA**]; much of the simpler situation with Noetherian hypotheses, which suffices for most applications, can be found in [**47**].

is called **cartesian** if it commutes, and the resulting morphism from X' to $Y' \times_Y X$ is an isomorphism. This agrees with the categorical notion of a cartesian diagram; in particular, X' is determined up to canonical isomorphism.

If $X \to Y$ is a family of some kind, then $X' = Y' \times_Y X \to Y'$ is called the **pullback** of the family for the morphism $Y' \to Y$. When $Y' = \mathrm{Spec}(\kappa(y))$, for y a point in Y, and $\kappa(y)$ the residue field of the local ring of Y at y, this pullback is the **fiber** of the family at y, and is denoted $f^{-1}(y)$. If Y is integral, with K the quotient field of its local rings, and $Y' = \mathrm{Spec}(K)$, the pullback is the **generic fiber**. When $Y' = \mathrm{Spec}(L)$, with L an algebraically closed field, the pullback is called a **geometric fiber** of the family.

For any morphism $f\colon X \to Y$, there is a canonical **diagonal** morphism $\Delta_f = (f,f)\colon X \to X \times_Y X$.

For any scheme X, there is a contravariant functor h_X from the category (Sch) of schemes to the category (Set) of sets, that takes a scheme S to the set $h_X(S) = \mathrm{Hom}(S,X)$ of morphisms from S to X. The elements of $h_X(S)$ are called S-valued points of X. For any scheme S, if $h_X(S)$ denotes the set of morphisms from S to X, there is a canonical bijection

$$h_{X\times_Z Y}(S) \leftrightarrow h_X(S) \times_{h_Z(S)} h_Y(S),$$

where the fibered product on the right is that of sets.

Schemes are often constructed by **recollement**, also called *gluing*, or *patching*. For this, one has a collection X_α of schemes, with an open subscheme $U_{\alpha\beta}$ of X_α for any pair α, β, so that $U_{\alpha\alpha} = X_\alpha$. In addition, one has isomorphisms $\vartheta_{\beta\alpha}$ of $U_{\alpha\beta}$ with $U_{\beta\alpha}$. These must satisfy the following compatibility condition: for any α, β, and γ, $\vartheta_{\beta\alpha}$ maps $U_{\alpha\beta} \cap U_{\alpha\gamma}$ isomorphically onto $U_{\beta\alpha} \cap U_{\beta\gamma}$, and the diagram

$$\begin{array}{ccccc}
U_{\alpha\beta}\cap U_{\alpha\gamma} & \xrightarrow{\vartheta_{\beta\alpha}} & U_{\beta\alpha}\cap U_{\beta\gamma} & = & U_{\beta\gamma}\cap U_{\beta\alpha} \\
\| & & & & \downarrow{\scriptstyle \vartheta_{\gamma\beta}} \\
U_{\alpha\gamma}\cap U_{\alpha\beta} & \xrightarrow[\vartheta_{\gamma\alpha}]{} & U_{\gamma\alpha}\cap U_{\gamma\beta} & = & U_{\gamma\beta}\cap U_{\gamma\alpha}
\end{array}$$

must commute. (It follows that $\vartheta_{\alpha\alpha}$ is the identity on X_α, and that $\vartheta_{\alpha\beta} \circ \vartheta_{\beta\alpha}$ is the identity on $U_{\alpha\beta}$.) Then there is a scheme X, with open embeddings $\varphi_\alpha\colon X_\alpha \to X$, such that: X is the union of the $\varphi(X_\alpha)$; for all α and β, $\varphi_\alpha(U_{\alpha\beta}) = \varphi_\alpha(X_\alpha) \cap \varphi_\beta(X_\beta)$; and $\varphi_\alpha = \varphi_\beta \circ \vartheta_{\beta\alpha}$ on $U_{\alpha\beta}$. The same construction works for any ringed spaces.

Let $\mathcal{C}$ be the category (Set) of schemes, or the category (Sch/Λ) of schemes over a base scheme Λ. A contravariant functor F from $\mathcal{C}$ to the category (Set) of sets is called a **sheaf** if, for every (Zariski) open covering $\{U_\alpha\}$ of a scheme X, the sequence

$$F(X) \to \prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \cap U_\beta)$$

is exact; that is, any element of $F(X)$ is determined by its restrictions to the open sets U_α, and a collection of elements in $F(U_\alpha)$ that agree on the overlaps $U_\alpha \cap U_\beta$ come from a unique element in $F(X)$.

A representable natural transformation $F \to G$ between contravariant functors from $\mathcal{C}$ to (Set) is called **open** if, for every scheme Z and natural transformation $h_Z \to G$, the functor $F \times_G h_Z$ is represented by a scheme Y, and the corresponding morphism from Y to Z is an open embedding.[6]

A collection of natural transformations $F_\alpha \to F$ is an **open covering** if each $F_\alpha \to F$ is representable and open, and, for any scheme Z and natural transformation $h_Z \to F$, the images of the schemes representing $F_\alpha \times_F h_Z$ in Z form an open covering of Z. With these definitions we have:

Proposition (Grothendieck's representability theorem). *Let F be a contravariant functor from $\mathcal{C}$ to* (Set). *Suppose there is a family F_α of subfunctors of F, such that each F_α is representable, and the collection $F_\alpha \to F$ is an open covering. Then F is representable.*

To prove this, if X_α represents F_α, the fibered products $F_\alpha \times_F F_\beta$ determine open coverings $U_{\alpha\beta}$ of X_α, together with isomorphisms $\vartheta_{\beta\alpha}$ of $U_{\alpha\beta}$ with $U_{\beta\alpha}$. One verifies, using triple fibered products $F_\alpha \times_F F_\beta \times_F F_\gamma$, that these isomorphisms satisfy the compatibility conditions of recollement, so that X is constructed by gluing these X_α. For details of this verification, see [**EGA** I′.0.4.5.4]. The same argument works in the category of ringed spaces.

2. Morphisms

A morphism $f\colon X \to Y$ of schemes is an **embedding**[7] if f factors into an isomorphism $X \to X'$ followed by the inclusion $X' \to Y$ of a subscheme X' of Y. It is an **open embedding** if X' is an open subscheme of Y, and a **closed embedding** if X' is a closed subscheme of Y. For a general morphism $f\colon X \to Y$ of schemes, the diagonal $\Delta_f\colon X \to X \times_Y X$ is an embedding ([**EGA** I.5.3.9, Err_{III}.10]).

A morphism $f\colon X \to Y$ is **locally of finite type** if for every x in X, with $y = f(x)$, there are affine neighborhoods $U \cong \mathrm{Spec}(B)$ of x and $V \cong \mathrm{Spec}(A)$ of y, with $f(U) \subset V$, such that the induced map $A = \Gamma(V, \mathcal{O}_Y) \to B = \Gamma(U, \mathcal{O}_U)$ makes B a finitely generated A-algebra; that is, $B \cong A[X_1, \dots X_n]/I$ for some ideal I. The morphism is **locally of finite presentation** if one can find such neighborhoods with B of finite presentation over A; that is, $B \cong A[X_1, \dots X_n]/I$, with $I = (F_1, \dots, F_m)$ for some polynomials $F_i \in A[X_1, \dots X_n]/I$. When Y is locally Noetherian, these two notions coincide. A morphism f is **of finite type** if every point of Y has an affine open neighborhood $V \cong \mathrm{Spec}(A)$ such that $f^{-1}(V)$ is covered by a finite number of affine open sets $U \cong \mathrm{Spec}(B)$ with B a finitely generated A-algebra. This implies that the same property holds for every affine open subset of Y ([**EGA** I.6.3]).

Most ordinary morphisms, such as those between algebraic varieties over a field, will be of finite type. However, morphisms like $\mathrm{Spec}(K) \to X$, where K is the function field of an integral scheme X, or morphisms like $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{Q})$, although not of finite type, are often useful.

[6] Such a definition makes sense for any property of a morphism which is preserved by arbitrary pullbacks and by composing on either side by an isomorphism; all of the properties of morphisms defined in the next section have this property.

[7] We avoid the word "immersion" for this notion, since that word has such a different meaning in differential geometry.

A scheme X is **quasi-compact** if its underlying space has the property that every covering by open sets has a finite subcover; equivalently, X can be covered by a finite number of affine open subsets. (Note that $\mathrm{Spec}(A)$ is quasi-compact for any A, whether Noetherian or not.) A morphism $f\colon X \to Y$ of schemes is **quasi-compact** if $f^{-1}(U)$ is quasi-compact for every affine open subset U of Y. It suffices in fact that this property holds for every U in one affine open covering of Y ([**EGA** I.6.6]). A morphism is of finite type exactly when it is locally of finite type and quasi- compact.

A morphism $f\colon X \to Y$ is **separated** if the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is a closed embedding; equivalently, the image of Δ_f is a closed subset of $X \times_Y X$. E.g., every morphism of affine schemes is separated. The **valuative criterion for separatedness** asserts that f is separated if and only if (i) the diagonal $X \to X \times_Y X$ is quasi-compact, and (ii) for any valuation ring R, with quotient field L, and any morphisms $\mathrm{Spec}(R) \to Y$ and $\mathrm{Spec}(L) \to X$ such that the diagram

$$\begin{array}{ccc} \mathrm{Spec}(L) & \longrightarrow & X \\ \big\downarrow & & \big\downarrow f \\ \mathrm{Spec}(R) & \longrightarrow & Y \end{array}$$

commutes, there is *at most* one morphism from $\mathrm{Spec}(R)$ to X making the whole diagram commute; this criterion asserts that the canonical map

$$\mathrm{Hom}_Y(\mathrm{Spec}(R), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(L), X)$$

is injective. When Y is locally Noetherian, one needs this test only when R is a discrete valuation ring. (See [**EGA** II.7.2.3] for these criteria.)

A morphism $f\colon X \to Y$ is **quasi-separated** if it satisfies the first condition (i) of the criterion for separatedness: the diagonal morphism $\Delta_f\colon X \to X \times_Y X$ is quasi-compact. Equivalently, for any affine open subsets U and V of X whose images are contained in an affine open subset of Y, the intersection $U \cap V$ is a finite union of affine open subsets.

A scheme X is called **separated** (resp. **quasi-separated**) if the morphism $X \to \mathrm{Spec}(\mathbb{Z})$ is separated (resp. quasi-separated). Every locally Noetherian scheme is quasi-separated.

A morphism $f\colon X \to Y$ is **proper** if it is separated, of finite type, and if, for any morphism $Y' \to Y$, the projection $X \times_Y Y' \to Y'$ is closed (i.e., the image of any closed subset is closed). The **valuative criterion for properness** asserts that f is proper if and only if (i) f is a separated morphism of finite type, and (ii) for any valuation ring R and morphisms as in the valuative criterion for separatedness, the canonical map $\mathrm{Hom}_Y(\mathrm{Spec}(R), X) \to \mathrm{Hom}_Y(\mathrm{Spec}(L), X)$ is surjective (and therefore bijective). When Y is locally Noetherian, it suffices to verify this criterion when R is a discrete valuation ring. (See [**EGA** II.7.3.8])

Recall that a homomorphism $A \to B$ of commutative rings is flat if the functor $M \to B \otimes_A M$ from A-modules to B-modules is (left) exact. A morphism $f\colon X \to Y$ of schemes is **flat** if for every point x in X, the local ring $\mathcal{O}_{x,X}$ is flat as a module over $\mathcal{O}_{y,Y}$. A morphism is **faithfully flat** if it is flat and surjective. For the morphism from

Spec(B) to Spec(A) coming from a homomorphism $A \to B$, this is equivalent to the flatness of B over A together with the assertion that the vanishing of $B \otimes_A M$ implies the vanishing of M, for any A-module M. A morphism is **fppf** if it is faithfully flat and locally of finite presentation. An important fact is that any fppf morphism is open, i.e., the image of any open set is open [**EGA** IV.2.4.6]. A morphism is **fpqc** if it is faithfully flat and quasi-compact.

A morphism $f\colon X \to Y$ is **unramified** if f is of locally of finite presentation and, for every x in X, with $y = f(x)$, one has $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$ and $\kappa(x)$ is a finite separable field extension of $\kappa(y)$. For f locally of finite presentation, this is equivalent to each of the following assertions: (i) the diagonal morphism $X \to X \times_Y X$ is an open embedding; (ii) the sheaf $\Omega^1_{X/Y}$ of relative differentials vanishes; (iii) for any nilpotent ideal I in a commutative ring Λ, and any morphism $\operatorname{Spec}(\Lambda/I) \to X$, there is *at most* one morphism from $\operatorname{Spec}(\Lambda)$ to X so that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Spec}(\Lambda/I) & \longrightarrow & X \\ \big\downarrow & \nearrow & \big\downarrow f \\ \operatorname{Spec}(\Lambda) & \longrightarrow & Y \end{array}$$

That is, the canonical map

$$\operatorname{Hom}_Y(\operatorname{Spec}(\Lambda), X) \to \operatorname{Hom}_Y(\operatorname{Spec}(\Lambda/I), X)$$

is injective.

A morphism $f\colon X \to Y$ is **étale** if it is unramified and flat. Equivalently, f is locally of finite presentation and, with Λ and I as above, one can always fill in the diagram uniquely: the canonical map $\operatorname{Hom}_Y(\operatorname{Spec}(\Lambda), X) \to \operatorname{Hom}_Y(\operatorname{Spec}(\Lambda/I), X)$ is bijective.

A morphism $f\colon X \to Y$ is **smooth** if it is locally of finite presentation, flat, and, for any morphism $\operatorname{Spec}(L) \to Y$, with L a field, the fiber $X \times_Y \operatorname{Spec}(L)$ is regular, i.e., all its local rings are regular local rings. Equivalently, f is locally of finite presentation, and, with Λ and I as above, the canonical map $\operatorname{Hom}_Y(\operatorname{Spec}(\Lambda), X) \to \operatorname{Hom}_Y(\operatorname{Spec}(\Lambda/I), X)$ is surjective. Equivalently, any point on X has a neighborhood U, mapped to an open subset V of Y, such that there is a commutative diagram

$$\begin{array}{ccc} U & \hookrightarrow & \operatorname{Spec}(A[X_1, \ldots, X_n]/(F_1, \ldots, F_m)) \\ \big\downarrow & & \big\downarrow \\ V & \hookrightarrow & \operatorname{Spec}(A) \end{array}$$

with the horizontal arrows open embeddings, and with $\operatorname{rank}(\partial F_i/\partial X_j) \equiv m$ on U. For f étale, one has the same local description but with $m = n$. A smooth morphism locally factors into a composition $U \to V \times \mathbb{A}^r \to V$, where the first map is étale and the second is the projection ([**EGA** IV.17.11.4]). Other characterizations and properties of unramified, étale, and smooth morphisms can be found in [**EGA** IV. §17].

A morphism is said to be **formally unramified**, resp. **formally étale**, resp. **formally smooth** if it satisfies the condition on liftings of morphisms $\operatorname{Spec}(\Lambda/I) \to X$ to $\operatorname{Spec}(\Lambda) \to X$ stated above for unramified, resp. étale, resp. smooth morphisms.

Smooth is equivalent to formally smooth and locally of finite presentation (and similarly for étale, unramified). We call particular attention to morphisms that are *formally unramified* and *locally of finite type*. This is a class of morphisms which naturally generalizes embeddings. By [**EGA** IV.17.2.1], a morphism is formally unramified if and only if it has trivial sheaf of relative differentials (see the section on differentials, below).

A morphism $f\colon X \to Y$ is **affine** if any point of Y has an affine open neighborhood V such that $f^{-1}(V)$ is an affine open subset of X. It follows that $f^{-1}(V)$ is affine for every affine open set V in Y. An affine morphism is separated and quasi-compact.

A morphism $f\colon X \to Y$ is **quasi-affine** if any point of Y has an affine open neighborhood V such that $f^{-1}(V)$ is isomorphic to a quasi-compact open subscheme of an affine scheme. Such a morphism is automatically separated and quasi-compact.

A morphism $f\colon X \to Y$ is **finite** if it is affine, and for any affine open $V \cong \mathrm{Spec}(A)$ of Y, $f^{-1}(V) \cong \mathrm{Spec}(B)$, with B finitely generated as an A-module. A morphism $f\colon X \to Y$ is **quasi-finite** if it is of finite type and each fiber $f^{-1}(y)$ is a finite set.

If Y is locally Noetherian, the following are equivalent: (i) f is finite; (ii) f is proper and affine; (iii) f is proper and quasi-finite [**EGA** III.4.4.2].

A morphism $f\colon X \to Y$ is **projective** if there is a quasi-coherent $\mathcal{O}_Y$-module $\mathcal{E}$ of finite type, such that f factors into a closed embedding $X \to \mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$ followed by the canonical projection from $\mathrm{Proj}(\mathrm{Sym}(\mathcal{E}))$ to Y.

An **invertible sheaf** is a locally free sheaf of rank one. An invertible sheaf $\mathcal{L}$ on a quasi-compact scheme X is **ample** if, for any coherent sheaf $\mathcal{F}$ on X, there is an integer n_0 such that, for all $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is generated by its sections. If $f\colon X \to Y$ is a quasi-compact morphism, an invertible sheaf $\mathcal{L}$ on X is **f-ample** if any point of Y has an affine open neighborhood U such that the restriction of $\mathcal{L}$ to $f^{-1}(U)$ is ample.

A morphism $f\colon X \to Y$ is **quasi-projective** if it is of finite type, and there is an f-ample invertible sheaf. If Y is quasi-compact (or its underlying space is Noetherian), this is equivalent to f factoring $X \to \mathrm{Proj}(\mathrm{Sym}(\mathcal{E})) \to Y$ as above, but with the first map only a locally closed embedding ([**EGA** II.5.3.2]). A projective morphism is proper and quasi-projective; the converse is true if the target scheme Y is quasi-compact or its underlying space is Noetherian ([**EGA** II.5.5.3]).

A morphism $f\colon X \to Y$ is an **epimorphism** if for any two morphisms g and h from Y to any scheme Z, $g \circ f = h \circ f$ implies $g = h$; that is, the canonical mapping from $\mathrm{Hom}(Y, Z)$ to $\mathrm{Hom}(X, Z)$ is always injective. It is an **effective epimorphism** if, whenever a morphism $\widetilde{g}\colon X \to Z$ is given such that $\widetilde{g} \circ p = \widetilde{g} \circ q$, where p and q are the two projections from $X \times_Y X$ to X, then there is a unique morphism $g\colon Y \to Z$ with $g \circ f = \widetilde{g}$; that is,

$$\mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z) \rightrightarrows \mathrm{Hom}(X \times_Y X, Z)$$

is exact. Any fppf or fpqc morphism is an effective epimorphism (see Appendix A).

A morphism $f\colon X \to Y$ is a **monomorphism** if for any morphisms g and h from any scheme S to X, $f \circ g = f \circ h$ implies $g = h$; that is, the map $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ is always injective. Equivalently, the diagonal map $\Delta_f\colon X \to X \times_Y S$ is an isomorphism ([**EGA** I.5.2.8]). In particular, f is separated; if it is locally of finite presentation, it is unramified. A morphism is **radicial** if, whenever $S = \mathrm{Spec}(L)$, for L a field, the map $\mathrm{Hom}(S, X) \to \mathrm{Hom}(S, Y)$ is always injective; equivalently, the map is injective on the underlying sets of points, and, for every point x of X, the field extension $\kappa(f(x)) \subset \kappa(x)$ is purely inseparable ([**EGA** I.3.5.8]). If f is locally of finite type, it is a monomorphism if and only if it is unramified and radicial ([**EGA** IV.17.2.6]).

Each of the properties of morphisms $f\colon X \to Y$ listed here is preserved by arbitary base change (except epimorphism, where this is an extra condition, termed *universal* epimorphism). That is, if f has the property, and $Y' \to Y$ is an arbitrary morphism, then the pullback $X \times_Y Y' \to Y'$ also has the property. If $f\colon X \to Y$ and $f'\colon X' \to Y'$ each have one of these properties, and there are morphisms from Y and Y' to a scheme S, then the fibered product $X \times_S X' \to Y \times_S Y'$ also has the property. If $f\colon X \to Y$ and $g\colon Y \to Z$ each have one of these properties, the composition $g \circ f$ also has the property; however, in the case of projective or quasi-projective morphisms, one must assume that Z is quasi-compact (or its underlying space is Noetherian). Any isomorphism satisfies all the properties. There are also results that say if $g\colon Y \to Z$ satisfies an appropriate condition (often separated suffices), if $g \circ f$ has the property, then f has the property. And, when $Y' \to Y$ is surjective and satisfies an appropriate condition (faithfully flat and quasi-compact is common), if $X \times_Y Y' \to Y'$ satisfies a property, then f will satisfy the property.[8]

3. Differentials

We recall and collect some basic facts about sheaves of differentials. We start with affine schemes (algebras) and the algebraic properties of modules of differentials. Then we pass to schemes and their sheaves of relative differentials.

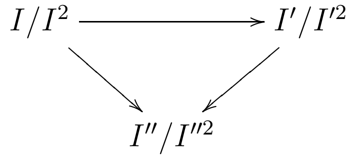
Attached to a surjective ring homomorphism $A \to B$, with kernel I, is the module I/I^2, which naturally has the structure of B-module. The associated sheaf is the conormal sheaf of $\mathrm{Spec}\, B$ in $\mathrm{Spec}\, A$. This construction applied to the relative diagonal gives the module of relative differentials.

Let B be an A-algebra. Then the **module of differentials** $\Omega_{B/A}$ is the sheaf I/I^2, where I is the kernel of the multiplication map $B \otimes_A B \to B$. It comes with a differential map $d\colon B \to \Omega_{B/A}$, defined by $df = 1 \otimes f - f \otimes 1$ for $f \in B$. This module satisfies the following universal property: for any B-module M and map $d'\colon B \to M$ which is additive, satisfies the Leibnitz rule $d'(fg) = f\,dg + g\,df$, and vanishes on A, there is a unique B-module homomorphism $\varphi\colon \Omega_{B/A} \to M$ such that $d' = \varphi \circ d$ (see [**47**, II.8]).

Considering, again, a surjective ring homomorphism $A \to B$ with kernel I, if $\varphi\colon A \to A'$ is an arbitrary ring homomorphism, we set $B' = B \otimes_A A'$, so $A' \to B'$ is also surjective. Let I' denote the kernel of $A' \to B'$. Then there is a morphism of B-modules $I/I^2 \to I'/I'^2$, induced by $f \mapsto \varphi(f)$. It is natural in the sense that given

[8] These and related results can be found in the following sections of [**EGA**], listed with the corresponding property: locally of finite type, IV.1.3.4; locally of finite presentation, IV.1.4.3; finite type, IV.1.5.4, quasi-compact, IV.1.1.2; separated, I.2.2; quasi- separated IV.1.2.2; proper II.5.4.2; flat IV.2.1; faithfully flat, IV.2.2.13; unramified, étale, and smooth, IV.17.3.3; affine, II.1.6.2; quasi-affine II.5.1.10; finite, II.6.1.5; quasi-finite, II.6.2.4; projective II.5.5.5; quasi-projective II.5.3.4; radicial I.3.5. General references for schemes and morphisms are: [**47**], [**24**], [**74**], [**68**], [**38**].

$\psi\colon A'' \to A'$ as well, with $B'' = B' \otimes_{A'} A'' \cong B \otimes_A A''$, then we get a commutative triangle



This is because the right-hand map sends $\varphi(f)$ to $\psi(\varphi(f)) = \psi \circ \varphi(f)$.

We have a morphism of B-modules $I/I^2 \to I'/I'^2$, or just as well, a morphism of B'-modules $(I/I^2) \otimes_B B' \to I'/I'^2$. This is an isomorphism when A' is flat over A, and also when $A \to B$ is left inverse to some $B \to A$ and the base change is induced by some base change on B ([**EGA** IV.16.2.2–3]): when φ is flat we have

$$(I/I^2) \otimes_B B' \cong I'/I'^2,$$

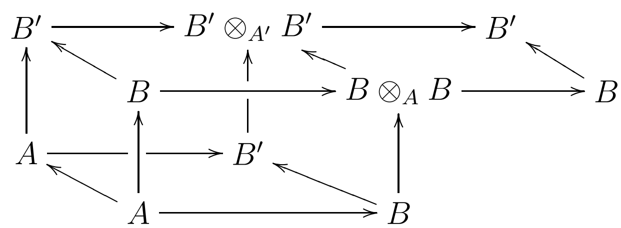
and when $B \to A$ is a ring homomorphism, having the given $A \to B$ as left inverse, and B' is a B-algebra, then with $A' = A \otimes_B B'$ (with the induced $\varphi\colon A \to A'$),

$$(I/I^2) \otimes_B B' \cong I'/I'^2.$$

If A' and B are A-algebras, and we set $B' = B \otimes_A A'$, then we have

$$\Omega_{B/A} \otimes_B B' \cong \Omega_{B'/A'}.$$

In other words, formation of $\Omega_{B/A}$ commutes with arbitrary base change. A reference is [**EGA** 0.20.5.5]; a hint to following the (terse) proof is to apply the second conormal sheaf isomorphism above to the diagram



There are two fundamental exact sequences on differentials. *First fundamental exact sequence:* ([**66**, Theorem 25.1], [**EGA** 0.20.5.7]) Given ring homomorphisms $A \to B$ and $B \to C$, this is the exact sequence

$$\Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0.$$

This is exact on the left as well when C is a formally smooth B-algebra. *Second fundamental exact sequence:* ([**66**, Theorem 25.2], [**EGA** 0.20.5.14]) As above, with $B \to C$ surjective with kernel J, this is the exact sequence

$$J/J^2 \to \Omega_{B/A} \otimes_B C \to \Omega_{C/A} \to 0.$$

If C is formally smooth over A then this is also left exact. These sequences are functorial [**EGA** 0.20.5.7.3], [**EGA** 0.20.5.11.3]. Explicitly, this means that if A' is an A-algebra, and we set $B' = B \otimes_A A'$ and $C' = C \otimes_A A'$ then the first fundamental exact sequence of the primed rings fits into a commutative diagram with that above, and the same is true for the second fundamental exact sequence when $B \to C$ is surjective.

If $f\colon X \to Y$ is a locally closed embedding of schemes, then f factors through some open subscheme $Y' \subset Y$ with $X \to Y'$ a closed embedding. Now there is a quasi-coherent sheaf of ideals $\mathcal{I}$ on Y' which defines the image of X as a subscheme. We call $f^*(\mathcal{I}/\mathcal{I}^2)$ the **conormal sheaf** to the embedding of X in Y. It is denoted $\mathcal{N}^*_{X/Y}$, or $\mathcal{N}^*_f$. If the restriction of f to an affine open subset $\operatorname{Spec} B$ of X is closed embedding to some affine open $\operatorname{Spec} A$ in Y then the restriction of $\mathcal{N}^*_{X/Y}$ to $\operatorname{Spec} B$ is the quasi-coherent sheaf associated to I/I^2, where I is the kernel of $A \to B$.

For a general map of schemes $X \to Y$, the relative diagonal $X \to X \times_Y X$ is an embedding The conormal sheaf to the relative diagonal is the **sheaf of relative differentials** of X over Y. It is denoted $\Omega_{X/Y}$, or Ω_f if f denotes the map of schemes.

Consider a cartesian diagram of schemes

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then we have an isomorphism

$$h^*\Omega_{X/Y} \cong \Omega_{X'/Y'}.$$

When f is a locally closed embedding, there is an induced morphism

$$h^*\mathcal{N}^*_{X/Y} \to \mathcal{N}^*_{X'/Y'}.$$

These are natural in the sense that in each case a morphism $Y'' \to Y'$ gives rise to a commutative triangle of sheaves on $X'' = X \times_Y Y''$ (this is immediate from the algebraic preliminaries for the morphism of conormal sheaves, and is established for the isomorphism of sheaves of differentials by extending the large diagram in Exercise **??**). The morphism of conormal sheaves is an isomorphism when g is flat.

The fundamental exact sequences, for schemes, read as follows. If $f\colon X \to Y$ and $g\colon Y \to Z$ are morphisms, then there is an exact sequence of sheaves on X

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

If f is formally smooth then this is left exact as well. When f is a locally closed embedding, we have an exact sequence of sheaves on X

$$\mathcal{N}^*_{X/Y} \to f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to 0.$$

This is also left exact when $g \circ f$ is formally smooth. These exact sequences are natural in the sense that if $Z' \to Z$ is a morphism and we set $X' = X \times_Z Z'$ and $Y' = Y \times_Z Z'$ then there are commutative diagrams relating the sequences for the primed and unprimed schemes.

The sheaf Ω_f, quasi-coherent in general, is of finite type when f is locally of finite type [**EGA** IV.16.3.9]. If f is formally smooth and locally of finite type then Ω_f is locally free of finite type [**EGA** IV.17.2.3(i)]. For a smooth morphism (i.e., one that is formally smooth and locally of finite presentation) we use the notation of relative dimension, a locally constant function on the source. This is the rank of Ω_f.

Let X and Y be schemes over a base scheme S. Consider the fiber product $X \times_S Y$, with projections p to X and q to Y. Then ([**EGA** IV.16.4.23])

$$p^*\Omega_{X/S} \otimes q^*\Omega_{Y/S} \cong \Omega_{X\times_S Y/S}.$$

The following is an important consequence of the second fundamental exact sequence, for schemes. Let $f\colon X \to Y$ be a smooth morphism. Let $s\colon Y \to X$ be a section of f (so $f \circ s = 1_Y$). Then

$$\mathcal{N}_s^* \cong s^*\Omega_{X/Y}.$$

In this situation we have, further, that s is a **regular embedding**, meaning that in a neighborhood of a point $y \in Y$ if we let r denote the relative dimension of f at $s(y)$ then $s(Y)$ is defined near $s(y)$ by r equations, forming a regular sequence in the local ring $\mathcal{O}_{s(y),X}$. This fact follows from [**EGA** IV.17.12.1].

4. Grothendieck topologies

A **Grothendieck topology** $\mathcal{T}$ on a category $\mathcal{S}$ consists of a set $\mathrm{Cov}(\mathcal{T})$ of families of maps $\{\varphi_\alpha\colon U_\alpha \to U\}_{\alpha\in\mathcal{A}}$, with each φ_α a morphism in $\mathcal{S}$; these families, called **coverings**, must satisfy the following conditions:

(1) If $\varphi\colon V \to U$ is an isomorphism in $\mathcal{S}$, then $\{\varphi\colon V \to U\}$ is a covering.

(2) If $\{U_\alpha \to U\}_{\alpha\in\mathcal{A}}$ is a covering, and $\{V_{\alpha\beta} \to U_\alpha\}_{\beta\in\mathcal{B}_\alpha}$ is a covering for each α, then the family $\{V_{\alpha\beta} \to U\}_{\alpha\in\mathcal{A},\beta\in\mathcal{B}_\alpha}$, obtained by composition, is a covering.

(3) If $\{U_\alpha \to U\}_{\alpha\in\mathcal{A}}$ is a covering, and $V \to U$ is any morphism in $\mathcal{S}$, each fiber product $U\,_\alpha\times_U V$ must exist in $\mathcal{S}$, and $\{U\,_\alpha\times_U V \to V\}_{\alpha\in\mathcal{A}}$ is a covering.

A category with a Grothendieck topology is called a **site**.

When $\mathcal{S}$ is the category (Top) of topological spaces, taking the coverings of a space U by a family of open subspaces U_α forms a Grothendieck topology. Similarly when $\mathcal{S}$ is any category of schemes which contains any open subscheme of any scheme in it, one has the **Zariski topology**, where a covering is a family of Zariski open subsets U_α of U, with each φ_α the inclusion of U_α in U, such that U is the union of these open sets. The examples of most importance in this text are the **étale topology** and the **smooth topology**; in these the morphisms φ_α are taken to be étale resp. smooth, with the condition that U is the union of the images of the U_α. Similarly one has the **flat topology**, also called the **fppf topology**, where one requires that the morphisms in a covering are faithfully flat and locally of finite presentation.

In each of these topologies, if $\{U_\alpha \to U\}$ is a covering, then the morphism $V = \coprod U_\alpha \to U$ is an fppf morphism, which means that descent (Appendix A) can be applied. Note also that if U is a disjoint union of open schemes U_α, the family $\{U_\alpha \to U\}$ is a covering in any of these topologies.

Although this text does not require any sophisticated knowledge of Grothendieck topologies, more can be found in [**3**] and [**68**].

5. Sheaves and base change

Our aim here is to describe the basic base change homomorphisms for sheaves, including compatibilities for successive base changes, for which we could not find complete references. We will see that these compatibilites follow formally from properties of adjoint functors that appear in Appendix B.

For a sheaf $\mathcal{F}$ on a space X, we denote its sections over an open subset U of X either by $\mathcal{F}(U)$ or $\Gamma(U, \mathcal{F})$. The **stalk** $\mathcal{F}_x$ of $\mathcal{F}$ at a point x in X is the direct limit $\varinjlim \mathcal{F}(U)$, as U varies over open neighborhoods of x. For any continuous map $f\colon X \to Y$ of topological spaces, and a sheaf $\mathcal{F}$ on X, there is a **pushforward sheaf** $f_*(\mathcal{F})$ on Y, whose sections over an open subset V of Y are defined by the formula

$$f_*(\mathcal{F})(V) \;=\; \mathcal{F}(f^{-1}(V)).$$

This pushforward is functorial: if also $g\colon Y \to Z$, then

$$(g \circ f)_*(\mathcal{F}) = g_*(f_*(\mathcal{F})).$$

If $\mathcal{G}$ is a sheaf on Y, there is a sheaf $f^{-1}(\mathcal{G})$, whose sections over an open U in X are defined to be those collections of elements $(s'_x)_{x \in U}$, with s'_x in the stalk $\mathcal{G}_{f(x)}$, such that for any x_0 in U there is a neighborhood V of $f(x_0)$ in Y, a section $s \in \mathcal{G}(V)$ and a neighborhood W of x_0 contained in $U \cap f^{-1}(V)$, such that s'_x is the germ defined by s at $f(x)$ for all x in W. This gives a functor from sheaves on Y to sheaves on X, which is a left adjoint to f_*. That is, for any sheaves $\mathcal{F}$ on X and $\mathcal{G}$ on Y, there is a canonical bijection

$$\mathrm{Hom}(f^{-1}(\mathcal{G}), \mathcal{F}) \leftrightarrow \mathrm{Hom}(\mathcal{G}, f_*(\mathcal{F})).$$

In fact, an element on each side of this display can be identified with a collection of maps from $\mathcal{G}(V)$ to $\mathcal{F}(U)$, for all open $U \subset X$ and $V \subset Y$ with $f(U) \subset V$, such that whenever $U' \subset U$ and $V' \subset V$, with $f(U') \subset V'$, the diagram

$$\begin{array}{ccc} \mathcal{G}(V) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{G}(V') & \longrightarrow & \mathcal{F}(U') \end{array}$$

commutes. This bijection is natural in morphisms of sheaves on X and Y, and makes f^{-1} a left adjoint of f_*, and f_* a right adjoint of f^{-1}, see [**EGA** 0.3.5, 0.3.7].

The same formula for the pushforward works when f is a morphism of ringed spaces, and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules, in which case $f_*(\mathcal{F})$ is a sheaf of $\mathcal{O}_Y$-modules. Here f_* defines a functor $f_*\colon \mathcal{S}(X) \to \mathcal{S}(Y)$ from the category $\mathcal{S}(X)$ of sheaves of $\mathcal{O}_X$-modules to the category $\mathcal{S}(Y)$ of sheaves of $\mathcal{O}_Y$-modules. A left adjoint to this functor f_* is denoted f^*; this is constructed to be the functor that takes a sheaf $\mathcal{G}$ of $\mathcal{O}_Y$-modules to the sheaf

$$f^*(\mathcal{G}) := \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} f^{-1}(\mathcal{G}).$$

(Here, to be precise, one should make a choice of this tensor product.) This gives a functor $f^*\colon \mathcal{S}(Y) \to \mathcal{S}(X)$ from sheaves of $\mathcal{O}_Y$-modules to sheaves of $\mathcal{O}_X$-modules, and one again has a canonical bijection

$$\mathrm{Hom}(f^*(\mathcal{G}), \mathcal{F}) \leftrightarrow \mathrm{Hom}(\mathcal{G}, f_*(\mathcal{F})),$$

making f_* and f^* adjoint functors ([**EGA** 0.3.5, 0.4.4]), [**47**, §II.5]).

Because of the choice of tensor product in the definition, if $g\colon Y \to Z$ is another morphism, and $\mathcal{H}$ is a sheaf of $\mathcal{O}_Z$-modules, then $(g \circ f)^*(\mathcal{H})$ is not strictly equal to $f^*(g^*(\mathcal{H}))$, but there is a canonical isomorphism between them.

This adjoint pair comes equipped with canonical natural transformations $\epsilon = \epsilon^f\colon 1_{\mathcal{S}(Y)} \Rightarrow f_* \circ f^*$, and $\delta = \delta^f\colon f^* \circ f_* \Rightarrow 1_{\mathcal{S}(X)}$. For a sheaf $\mathcal{G}$ on Y we have a canonical morphism $\epsilon\colon \mathcal{G} \to f_*(f^*(\mathcal{G}))$, functorial in $\mathcal{G}$; and for a sheaf $\mathcal{F}$ on X, we have a canonical morphism $\delta\colon f^*(f_*(\mathcal{F})) \to \mathcal{F}$, functorial in $\mathcal{F}$. Explicitly, a section of $\mathcal{G}$ on an open V in Y determines a section of $f^*(\mathcal{G})$ on $f^{-1}(V)$, and hence a section of $f_*(f^*(\mathcal{G}))$ on V. A section of $f^*(f_*(\mathcal{F}))$ on an open U of X determines an element of the stalk $\mathcal{F}_x$ at all x in U, and these come from a section of $\mathcal{F}$ on U.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is **quasi-coherent** if, for all x in X, there is a neighborhood U of x and a presentation $\mathcal{O}_U^{(I)} \to \mathcal{O}_U^{(J)} \to \mathcal{F} \to 0$, for some (not necessarily finite) index sets I and J. If $\mathcal{G}$ is quasi-coherent on Y, then $f^*(\mathcal{G})$ is always quasi-coherent on X. If f is a quasi-compact and quasi-separated morphism of schemes, and $\mathcal{F}$ is quasi-coherent on X, then $f_*(\mathcal{F})$ is quasi-coherent on Y (see [**EGA** I.9.2.1]). With these hypotheses, f_* and f^* are adjoint functors between quasi-coherent sheaves on X and quasi-coherent sheaves on Y.

A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is of **finite type** if any point has an open neighborhood U on which there is a surjection $\mathcal{O}_U^n \to \mathcal{F}|_U \to 0$ for some integer n, The sheaf is **coherent** if, in addition, for all open subsets U of X, the kernel of any homomorphism $\mathcal{O}_U^m \to \mathcal{F}$ of $\mathcal{O}_U$-modules is of finite type. If a scheme X is **locally Noetherian**, i.e., it has a covering by open subschemes isomorphic to the Spec's of Noetherian rings, then $\mathcal{O}_X$ is coherent. For any morphism $f\colon X \to Y$, if $\mathcal{G}$ is coherent on Y, and if $\mathcal{O}_X$ is coherent on X, then $f^*(\mathcal{G})$ is coherent on X ([**EGA** 0.5.3]). If $f\colon X \to Y$ is a proper morphism, with Y locally Noetherian, and $\mathcal{F}$ is a coherent sheaf on X, then $f_*(\mathcal{F})$ is a coherent sheaf on Y; in fact, all the higher direct images $R^n f_*(\mathcal{F})$ are coherent ([**EGA** III.3.2.1]).

Now consider a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Z \end{array}$$

of ringed spaces. There is an associated **base change** map

$$p^*(f_*(\mathcal{F})) \to g_*(q^*(\mathcal{F}))$$

for any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules.[9] This is a formal consequence of adunction (Section B.3). To construct this base change map, consider the diagram

$$\begin{array}{ccc} \mathcal{S}(W) & \xrightarrow{g_*} & \mathcal{S}(Y) \\ {\scriptstyle q_*}\big\downarrow & & \big\downarrow{\scriptstyle p_*} \\ \mathcal{S}(X) & \xrightarrow[f_*]{} & \mathcal{S}(Z) \end{array}$$

This diagram strictly commutes: $f_* \circ q_* = (f \circ q)_* = (p \circ g)_* = p_* \circ g_*$. Therefore we can take $\alpha\colon f_* \circ q_* \Rightarrow p_* \circ g_*$ to be the identity map. From Definition B.21, this gives a base change natural transformation (2-morphism) $c = c_\alpha$ from $p^* \circ f_*$ to $g_* \circ q^*$. The base change transformation $p^* \circ f_* \Rightarrow g_* \circ q^*$ is chararacterized either by the fact that its adjoint with respect to p is the transformation $f_* = f_* \circ 1_{\mathcal{S}(X)} \overset{\epsilon^q}{\Rightarrow} f_* \circ q_* \circ q^* = p_* \circ g_* \circ q^*$, or that its adjoint with respect to g is the composite $g^* \circ p^* \circ f_* \overset{\alpha'}{\Rightarrow} q^* \circ f^* \circ f_* \overset{\delta f}{\Rightarrow} q^*$ (by Exercise B.43(3)).[10]

In this setting, these formal adjoint constructions can be made explicit. If $U \subset Y$ and $U' \subset Y'$ are open subsets with $p(U') \subset U$, a section s in $(f_*\mathcal{F})(U)$ determines a section s' in $p^*(f_*\mathcal{F})(U')$, and also a section s'' in $(g_*(q^*(\mathcal{F})))(U') = (q^*(\mathcal{F}))(q^{-1}(U'))$. Show that the base change map c takes s' to s'', and show that c is determined by this property. The corresponding $\alpha'\colon g^* \circ p^* \overset{\sim}{\Rightarrow} q^* \circ f^*$ agrees with the composition of the canonical isomorphisms $g^* \circ p^* \cong (p \circ g)^* = (f \circ q)^* \cong q^* \circ f^*$ (cf. [**EGA** 0.3.5.5]).

Consider a commutative diagram of ringed spaces,

$$\begin{array}{ccccc} U & \xrightarrow{i} & W & \xrightarrow{g} & Y \\ {\scriptstyle r}\big\downarrow & \overset{\beta}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle q} & \overset{\alpha}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle p} \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Z \end{array}$$

with two commuting squares labeled with α and β, and label the outside square with γ:

$$\begin{array}{ccc} U & \xrightarrow{g\circ i} & Y \\ {\scriptstyle r}\big\downarrow & \overset{\gamma}{\searrow\!\!\searrow} & \big\downarrow{\scriptstyle p} \\ V & \xrightarrow[f\circ h]{} & Z \end{array}$$

It follows from Exercise B.44 that the following diagram commutes:

$$\begin{array}{ccccc} p * \circ f_* \circ h_* & = & p^* \circ (f \circ h)_* & & \\ {\scriptstyle c_\alpha}\Big\Downarrow & & & \overset{c_\gamma}{\searrow\!\!\searrow} & \\ g_* \circ q^* \circ h_* & \underset{c_\beta}{\Longrightarrow} & g_* \circ i_* \circ r^* & = & (g \circ i)_* \circ r^* \end{array}$$

[9]The same works for topological spaces, replacing pullbacks g^* by g^{-1}.

[10]See [**42**, XII.4, XVII.2.1] for a discussion about this point.

One has also the opposite base change maps $c'_\alpha\colon f^*\circ p_* \Rightarrow q_*\circ g^*$, $c'_\beta\colon h^*\circ q_* \Rightarrow r_*\circ i^*$, and $c'_\gamma\colon (f\circ h)^*\circ p_* \Rightarrow r_*\circ(g\circ i)^*$. The same Exercise B.44 gives a commutative diagram

$$\begin{array}{ccccc}
h^*\circ f^*\circ p_* & \overset{\sim}{=\!=} & (f\circ h)^*\circ p_* & & \\
\Downarrow{\scriptstyle c'_\alpha} & & & \overset{c'_\gamma}{\searrow} & \\
h^*\circ q_*\circ g^* & \underset{c'_\beta}{\Longrightarrow} & r_*\circ i^*\circ g^* & \overset{\sim}{=\!=} & r_*\circ(g\circ i)^*
\end{array}$$

(cf. [**42**, XII.4.4]).

From Exercise B.43(1), we get commutative diagrams

$$\begin{array}{ccc}
p^* & \overset{\epsilon^f}{\Longrightarrow} & p^*\,f_*\,f^* \\
{\scriptstyle \epsilon^g}\Downarrow & & \Downarrow{\scriptstyle c} \\
g_*\,g^*\,p^* & \underset{\alpha'}{\Longrightarrow} & g_*\,q^*\,f^*
\end{array}
\qquad
\begin{array}{ccc}
g^*\,p^*\,f_* & \overset{\alpha'}{\Longrightarrow} & q^*\,f^*\,f_* \\
\Downarrow{\scriptstyle c} & & \Downarrow{\scriptstyle \delta^f} \\
g^*\,g_*\,q^* & \underset{\delta^g}{\Longrightarrow} & q^*
\end{array}
\qquad
\begin{array}{ccc}
f_* & \overset{\epsilon^p}{\Longrightarrow} & p_*\,p^*\,f_* \\
{\scriptstyle \epsilon^q}\Downarrow & & \Downarrow{\scriptstyle c} \\
f_*\,q_*\,q^* & =\!=\!= & p_*\,g_*\,q^*
\end{array}$$

The same adjoint formalism applies in the context of sheaves on arbitrary sites. It can also be applied with higher direct images. To see this, note that if $f\colon X\to Y$ and $g\colon Y\to Z$ are mappings, the Leray spectral sequence gives (edge homomorphism) mappings

$$R^n g_*(f_*\mathcal{F}) \to R^n(g\circ f)_*(\mathcal{F}) \to g_*(R^n f_*(\mathcal{F}))$$

(cf. [**EGA** III.12.2.5]). In particular, given a commutative diagram as above, and any $n\geq 0$, one has a natural transformation $\alpha\colon R^n f_*\circ q_* \Rightarrow p_*\circ R^n g_*$, given by

$$R^n f_*(q_*(\mathcal{F})) \to R^n(f\circ q)_*(\mathcal{F}) = R^n(p\circ g)_*(\mathcal{F}) \to p_*(R^n g_*(\mathcal{F})).$$

By the formal properties of adjoints, this determines a natural transformation c_α from $p^*\circ R^n f_*$ to $R^n g_*\circ q^*$. In particular we have homomorphisms

$$p^*(R^n f_*(\mathcal{F})) \to R^n g_*(q^*(\mathcal{F})),$$

which are natural in $\mathcal{F}$. One has the same compatibility as before, when two commutative diagrams are pasted together, again by formal properties of adjoint functors.[11]

The same formalism applies when one has adjoint functors Rf_* and Lf^* on derived categories (e.g. [**46**], Cor. 5.11), giving natural base change maps

$$(Lp^*)\circ(Rf_*)(\mathcal{F}^\cdot) \;\to\; (Rg_*)\circ(Lq^*)(\mathcal{F}^\cdot),$$

with the corresponding compatibilites when two commutative diagrams are combined.

[11] These base change maps agree with those constructed under additional hypotheses in [**EGA** III.1.4.15], [**EGA** IV.1.7.21], and [**47**, §III.9.3].