APPENDIX B

Categories and 2-categories

I do not believe in categories of any kind. *Duke Ellington*

We begin this appendix by reviewing some basic notions about categories. The second section defines and proves basic properties of 2-categories. These are applied in Section 3 to the study of adjoint functors. The fourth section has the main theorem, which spells out the appropriate notion of equivalence for 2-categories. Most of these notions and results are known in some form in general category theory. We have tried to present them in more concrete terms than usual, and hope that this, and a deficiency of references, will not offend category theorists. We expect geometers will find the going abstract enough; for a first reading, it should suffice to concentrate on the definitions, examples, and statements of the propositions.

In the last section we make a few remarks about set theoretic foundations, and the axiom of choice, which is used freely in the text. These are not designed to put us in any axiomatic set-theoretical framework, but rather to explain why we avoid doing this.

1. Categories

A **category** $\mathcal{C}$ has **objects** and **morphisms**, also called **maps** or **mappings** or **arrows**. To each morphism is associated two objects, its **source** and its **target**. We write $f\colon X \to Y$ to mean that f is a morphism with the object X as its source and the object Y as its target, and we say that f is a morphism **from** X **to** Y.[1] For any morphism f from X to Y, and any morphism g from Y to Z, there must be a morphism from X to Z, called the **composite** of f and g, and denoted $g \circ f$ or sometimes simply gf. The following properties must be satisfied:

(a) For any object X there is a morphism $1_X\colon X \to X$ such that $f \circ 1_X = f$ for all $f\colon X \to Y$ and $1_X \circ g = g$ for all $g\colon Y \to X$.

(b) For any $f\colon X \to Y$, $g\colon Y \to Z$, and $h\colon Z \to W$, $h \circ (g \circ f) = (h \circ g) \circ f$.

Exercise B.1. Identity maps, if they exist, are unique.

A map $f\colon X \to Y$ is an **isomorphism** if there is a map $f^{-1}\colon Y \to X$ such that $f^{-1} \circ f = 1_X$ and $f \circ f^{-1} = 1_Y$.

[1] Although the notation $f\colon X \to Y$ is suggested by the functional notation of set theory, it does *not* mean that f assigns elements of Y to elements of X. In the category of schemes, for example, a morphism is much more than a function on underlying sets.

Exercise B.2. (1) An inverse, if it exists, is unique. (2) If $f\colon X \to Y$ is an isomorphism, and $g\colon Y \to Z$ is an isomorphism, then $g \circ f$ is an isomorphism, with inverse $f^{-1} \circ g^{-1}$.

A **subcategory** $\mathcal{C}'$ of a category $\mathcal{C}$ consists of some of the objects of $\mathcal{C}$ and some of the morphisms of $\mathcal{C}$, such that: (a) the source and target of any morphism in $\mathcal{C}'$ is in $\mathcal{C}'$; (b) if $f\colon X \to Y$ and $g\colon Y \to Z$ are in $\mathcal{C}'$, then $g \circ f$ is also in $\mathcal{C}'$; and (c) if an object X is in $\mathcal{C}'$, then 1_X is also in $\mathcal{C}'$. It follows that $\mathcal{C}'$ forms a category.

If $\mathcal{C}$ and $\mathcal{D}$ are categories, a (covariant) **functor** F from $\mathcal{C}$ to $\mathcal{D}$ assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, and to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$, such that: (a) if $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{C}$, then $F(g \circ f) = F(g) \circ F(f)$; (b) $F(1_X) = 1_{F(X)}$ for all objects X of $\mathcal{C}$. We write $F\colon \mathcal{C} \to \mathcal{D}$ to mean that F is a functor from $\mathcal{C}$ to $\mathcal{D}$.

Exercise B.3. (1) If $f\colon X \to Y$ is an isomorphism, then $F(f)$ is an isomorphism, with inverse $F(f^{-1})$. (2) In the definition of functor, the property that $F(1_X) = 1_{F(X)}$ could be replaced by the weaker property that $F(1_X)$ is an isomorphism, or that it has a left or a right inverse.

If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are functors, their **composite**, denoted $G \circ F$ or GF, is the functor from $\mathcal{C}$ to $\mathcal{E}$ defined by $G \circ F(X) = G(F(X))$ and $G \circ F(f) = G(F(f))$. With this composition law, the categories form a category, denoted (Cat).

If F and G are functors from $\mathcal{C}$ to $\mathcal{D}$, a **natural transformation** θ from F to G assigns to each object X in $\mathcal{C}$ a morphism θ_X from $F(X)$ to $G(X)$ in $\mathcal{D}$, such that for any morphism $f\colon X \to Y$ in $\mathcal{C}$, $G(f) \circ \theta_X = \theta_Y \circ F(f)$, i.e., the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \theta_X}\downarrow & & \downarrow{\scriptstyle \theta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

commutes. The notation $\theta\colon F \Rightarrow G$ is used to indicate that θ is a natural transformation from F to G. It is a **natural isomorphism** if each θ_X is an isomorphism, in which case one writes $\theta\colon F \overset{\sim}{\Rightarrow} G$.

If F, G, and H are functors from $\mathcal{C}$ to $\mathcal{D}$, two natural transformations θ from F to G and η from G to H can be composed, giving a natural transformation $\eta \circ \theta$ from F to H. This is defined by setting $(\eta \circ \theta)_X = \eta_X \circ \theta_X$.

Exercise B.4. (1) For fixed categories $\mathcal{C}$ and $\mathcal{D}$, there is a category $\mathrm{HOM}(\mathcal{C}, \mathcal{D})$ (or $\mathrm{HOM}_{(\mathrm{Cat})}(\mathcal{C}, \mathcal{D})$) with objects the functors from $\mathcal{C}$ to $\mathcal{D}$, and with arrows from F to G the natural transformations. (2) If θ is a natural isomorphism from F to G, then θ^{-1}, defined by $(\theta^{-1})_X = (\theta_X)^{-1}$, is a natural isomorphism from G to F, with $\theta^{-1} \circ \theta = 1_F$ and $\theta \circ \theta^{-1} = 1_G$.

A functor $F\colon \mathcal{C} \to \mathcal{D}$ is a **strict isomorphism** if there is a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $G \circ F$ and $F \circ G$ are the identity functors $1_{\mathcal{C}}$ on $\mathcal{C}$ and $1_{\mathcal{D}}$ on $\mathcal{D}$.

A functor $F: \mathcal{C} \to \mathcal{D}$ is an **equivalence** of categories if there is a functor $G: \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms θ from $G \circ F$ to $1_{\mathcal{C}}$ and η from $F \circ G$ to $1_{\mathcal{D}}$. (Note that only the existence of G, θ, and η is required, and they need not be unique.)

A functor $F: \mathcal{C} \to \mathcal{D}$ is called **faithful** if for any morphisms $f: X \to Y$ and $g: X \to Y$ in $\mathcal{C}$, the equality of $F(f)$ and $F(g)$ implies the equality of f and g. A functor F is called **full** if, for any objects X and Y of $\mathcal{C}$, any morphism from $F(X)$ to $F(Y)$ in $\mathcal{D}$ has the form $F(f)$ for some $f: X \to Y$ in $\mathcal{C}$. A functor $F: \mathcal{C} \to \mathcal{D}$ is **essentially surjective** if, for every object X in $\mathcal{D}$, there is an object P in $\mathcal{C}$ and an isomorphism from $F(P)$ to X in $\mathcal{D}$.

The inclusion of a subcategory $\mathcal{C}'$ in a category $\mathcal{C}$ is always a faithful functor. If $\mathcal{C}'$ is obtained by choosing some of the objects of $\mathcal{C}$, and all morphisms between them, this inclusion is also full, and $\mathcal{C}'$ is called a **full subcategory**.

Exercise B.5. Suppose F and G are naturally isomorphic functors. Then F is faithful (resp. full, resp. essentially surjective) if and only if G is faithful (resp. full, resp. essentially surjective).

Exercise B.6. If $F: \mathcal{C} \to \mathcal{D}$ is full and faithful, and $f: X \to Y$ is a morphism in $\mathcal{C}$, show that f is an isomorphism if and only if $F(f)$ is an isomorphism.

Proposition B.1. *A functor is an equivalence of categories if and only if it is full, faithful, and essentially surjective.*

Proof. We sketch the proof of the implication $\Leftarrow$. Suppose $F: \mathcal{C} \to \mathcal{D}$ is the functor. For each object X of $\mathcal{D}$, choose (by an appropriate axiom of choice if necessary, cf. Section 5) an object $G(X)$ of $\mathcal{C}$ and an isomorphism $\eta_X: F(G(X)) \to X$ in $\mathcal{D}$. For a morphism $f: X \to Y$ in $\mathcal{D}$, there is a unique morphism $G(f): G(X) \to G(Y)$ in $\mathcal{C}$ such that $F(G(f)) = {\eta_Y}^{-1} \circ f \circ \eta_X$. Verify that G is a functor. For an object P of $\mathcal{C}$, define $\theta_P: G(F(P)) \to P$ to be the morphism such that $F(\theta_P) = \eta_{F(P)}$, and verify that θ and η are natural isomorphisms. □

Exercise B.7. Complete the proof of this proposition.

Exercise B.8. Show that a functor $F: \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if there is a functor $G: \mathcal{D} \to \mathcal{C}$ and natural isomorphisms θ from $G \circ F$ to $1_{\mathcal{C}}$ and η from $F \circ G$ to $1_{\mathcal{D}}$ such that $F(\theta_P) = \eta_{F(P)}$ for all objects P in $\mathcal{C}$ and $G(\eta_X) = \theta_{G(X)}$ for all objects X in $\mathcal{D}$. In this case the data $(F, G, \theta^{-1}, \eta)$ is what is called an *adjoint equivalence*, cf. [**65**, §IV.4] and Section 3.

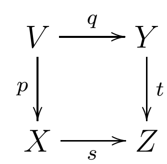
Exercise B.9. Let $F: \mathcal{C} \to \mathcal{D}$ and $G: \mathcal{D} \to \mathcal{E}$ be functors. (1) F and G faithful (resp. full, resp. essentially surjective) imply GF faithful (resp. full, resp. essentially surjective). (2) GF faithful implies F faithful; GF essentially surjective implies G essentially surjective; GF full and F essentially surjective implies G full; GF full and G full and faithful implies F full. (3) If GF is an equivalence of categories, and either F is essentially surjective or G is full and faithful, then F and G are both equivalences of categories.

Example B.2. A full subcategory $\mathcal{C}'$ of $\mathcal{C}$ is a **skeleton** of $\mathcal{C}$ if every object of $\mathcal{C}$ is isomorphic to exactly one object of $\mathcal{C}'$. The inclusion $\mathcal{C}' \to \mathcal{C}$ is then an equivalence of categories. For any category $\mathcal{C}$, the choice of one object from each isomorphism class of objects determines a skeleton $\mathcal{C}'$. For example, if $\mathcal{C}$ is the category of finite nonempty sets, the full subcategory whose objects are the sets $\{1, \ldots, n\}$ for $n \geq 1$ is a skeleton of $\mathcal{C}$.

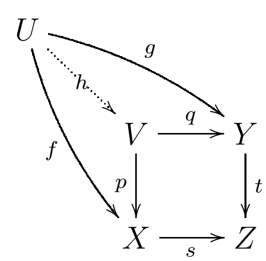
EXAMPLE B.3. The **product** $\mathcal{C} \times \mathcal{D}$ of two categories $\mathcal{C}$ and $\mathcal{D}$ is the category whose objects are pairs (X, Y) of objects X of $\mathcal{C}$, Y of $\mathcal{D}$; a morphism $(f, g)\colon (X, Y) \to (X', Y')$ is a pair of morphisms $f\colon X \to X'$ in $\mathcal{C}$ and $g\colon Y \to Y'$ in $\mathcal{D}$, with composition induced by that in each category. One constructs similarly a product of any number of categories.

The **opposite** category $\mathcal{C}^{\mathrm{op}}$ of a category $\mathcal{C}$ is obtained by reversing all the arrows of $\mathcal{C}$. A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor F from $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. This assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, and to each morphism $f\colon X \to Y$ of $\mathcal{C}$ a morphism $F(f)\colon F(Y) \to F(X)$. These satisfy: if $f\colon X \to Y$ and $g\colon Y \to Z$, then $F(g \circ f) = F(f) \circ F(g)$, as well as $F(1_X) = 1_{F(X)}$ for all objects X.

DEFINITION B.4. A commutative square



of objects and morphisms in a category $\mathcal{C}$ is called **cartesian** if it satisfies the following universal property. For any object U and morphisms $f\colon U \to X$ and $g\colon U \to Y$ such that $sf = tg$, there is a unique morphism $h\colon U \to V$ such that $ph = f$ and $qf = g$:



It follows that V is unique up to canonical isomorphism: if V', $p'\colon V' \to X$, $q'\colon V' \to Y$ also satisfy the universal property, there is a unique isomorphism $\vartheta\colon V' \to V$ such that $p' = p\vartheta$ and $q' = q\vartheta$.

If the diagram is cartesian, one writes $V = X \times_Z Y$, and V is called the (or a) **fibered product** of X and Y over Z. If the morphisms s and t need to be specified, one writes $V = X \,{}_s\!\times_t Y$ or $V = X \,{}_s\!\times_{Z,t} Y$. The morphism $h\colon U \to X \times_Z Y$ determined by f and g is usually denoted (f, g). The projection $X \times_Z Y \to X$ is often called the **pullback** of the morphism $Y \to Z$ by $s\colon X \to Z$.

If the category $\mathcal{C}$ has a final object $\bullet$ (so each object of $\mathcal{C}$ has a unique morphism to $\bullet$) then the fibered product $X \times_\bullet Y$ is called the **product** of X and Y, and denoted $X \times Y$.

Exercise B.10. Construct fibered products for arbitrary morphisms $s\colon X \to Z$ and $t\colon Y \to Z$ in the category (Set) of sets and the category (Top) of topological spaces.

Exercise B.11. (1) Given morphisms $s\colon X \to Z$, $t\colon Y \to Z$, $s'\colon X' \to Z'$, $t'\colon Y' \to Z'$, and morphisms $f\colon X' \to X$, $g\colon Y' \to Y$, $h\colon Z' \to Z$, with $sf = hs'$ and $tg = ht'$, construct a canonical morphism $X' \times_{Z'} Y' \to X \times_Z Y$, whenever these fibered products exist. (2) For any morphism $f\colon X \to Y$, construct a canonical morphism $X \to X \times_Y X$, whenever this fibered product exists; it is called the **diagonal** morphism.

Exercise B.12. (1) For any morphism $s\colon X \to Y$, the fibered product $X \times_Y Y = X\ {}_s{\times}_{Y,1_Y} Y$ exists and is canonically isomorphic to X. (2) There is a canonical isomorphism of $X\ {}_s{\times}_{Z,t} Y$ with $Y\ {}_t{\times}_{Z,s} X$, with one existing if and only if the other exists. (3) Suppose $s\colon X \to Z$, $t\colon Y \to Z$, $u\colon Y \to W$, and $v\colon V \to W$ are given, and $X \times_Z Y$ and $Y \times_W V$ exist. If one of the fibered products $(X \times_Z Y) \times_Y (Y \times_W V)$, $(X \times_Z Y) \times_W V$ or $X \times_Z (Y \times_W V)$ exists, then all exist and are canonically isomorphic. This fibered product is also denoted $X \times_Z Y \times_W V$; it is characterized by a universal property for triples of morphisms $f\colon U \to X$, $g\colon U \to Y$, and $h\colon U \to V$ such that $sf = tg$ and $ug = vh$: there is a unique morphism $(f, g, h)\colon U \to X \times_Z Y \times_W V$ such that f, g, and h are recovered by composing (f, g, h) with the projections to the three factors. (4) Suppose $s\colon X \to Y$, $t\colon Y \to Z$, and $f\colon W \to Z$ are morphisms, and $Y\ {}_t{\times}_{Z,f} W$ exists. Then $X\ {}_{ts}{\times}_{Z,f} W$ exists if and only if $X \times_Y (Y \times_Z W)$ exists, and then they are canonically isomorphic. (5) Suppose morphisms $X \to Z$, $Y \to Z$, and $Z \to T$ are given, and $X \times_T Y$ and $Z \times_T Z$ exist. Then $X \times_Z Y$ exists if and only if $(X \times_T Y) \times_{Z \times_T Z} Z$ exists, and then they are canonically isomorphic; here $X \times_T Y \to Z \times_T Z$ is the canonical map, and $Z \to Z \times_T Z$ the diagonal map, of the preceding exercise. In particular, if $\mathcal{C}$ has a final object, there is a canonical isomorphism $X \times_Z Y \cong (X \times Y) \times_{Z \times Z} Z$, whenever these fibered products exist.

For an object X in a category $\mathcal{C}$, define a contravariant functor h_X from $\mathcal{C}$ to the category (Set) of sets, that takes an object S to the set $h_X(S) = \mathrm{Hom}(S, X)$ of morphisms from S to X, and takes a morphism $u\colon T \to S$ to the mapping $h_X(u)\colon h_X(T) \to h_X(S)$ which sends $g\colon S \to X$ to $g \circ u\colon T \to X$. The elements of $h_X(S)$ are called S-**valued points of** X.

Exercise B.13. For any functor $H\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$, any object ζ in $H(X)$ determines a natural transformation from h_X to H; this assigns to an object S of $\mathcal{C}$ the map from $h_X(S)$ to $H(S)$ that takes $g\colon S \to X$ to $H(g)(\zeta)$. Show that every natural transformation from h_X to H arises in this way from a unique ζ in $H(X)$.

Any morphism $f\colon X \to Y$ in $\mathcal{C}$ determines a mapping from $h_X(S)$ to $h_Y(S)$ that takes $g\colon S \to X$ to $f \circ g\colon S \to Y$. This determines a covariant functor

$$\mathcal{C} \to \mathrm{HOM}(\mathcal{C}^{\mathrm{op}}, (\mathrm{Set})).$$

Exercise B.14. Show that this functor is full and faithful.

A functor $H\colon \mathcal{C}^{\mathrm{op}} \to (\mathrm{Set})$ is **representable by an object** X of $\mathcal{C}$ if one has a natural isomorphism between h_X and H. This is given by an element ζ in $H(X)$ such that, for all S, the map $h_X(S) \to H$ that takes $g\colon S \to X$ to $H(g)(\zeta)$ is a bijection. (Note that one must specify both X and ζ to represent H.) By Exercise B.14, the object X that represents H is determined up to canonical isomorphism. This combination of ideas is known as *Yoneda's Lemma*.

If $F \to G$ and $H \to G$ are natural transformations between functors from $\mathcal{C}^{\mathrm{op}}$ to (Set), there is a **fibered product** $F \times_G H$, which takes an object S of $\mathcal{C}$ to the set

$$(F \times_G H)(S) \,=\, F(S) \times_{G(S)} H(S)$$

of pairs of elements in $F(S)$ and $H(S)$ with the same image in $G(S)$. A morphism $u\colon T \to S$ in $\mathcal{C}$ is sent to the map from $F(S) \times_{G(S)} H(S)$ to $F(T) \times_{G(T)} H(T)$ determined by $F(u)$ and $H(u)$. The fibered product $F \times_G H$ is a contravariant functor from $\mathcal{C}$ to (Set). It comes equipped with natural transformations (called projections) from $F \times_G H$ to F and to H; it is a fibered product in the category of contravariant functors from $\mathcal{C}$ to (Set).

Exercise B.15. A commutative diagram as in Definition B.4 is cartesian if and only if, for every object S in $\mathcal{C}$, the corresponding diagram of S-valued points is a cartesian diagram in the category of sets. That is, the map

$$h_V(S) \;\to\; h_X(S) \times_{h_Z(S)} h_Y(S)$$

is a bijection. Equivalently, the canonical natural transformation from h_V to $h_X \times_{h_Z} h_Y$ is a natural isomorphism.

A natural transformation $F \to G$ between contravariant functors from $\mathcal{C}$ to (Set) is called **representable** if, for every object X in $\mathcal{C}$ and natural transformation $h_X \to G$, the fibered product $F \times_G h_X$ is representable. If Y is an object representing $F \times_G h_X$, the projection from $F \times_G h_X$ to h_X determines a morphism from Y to X in $\mathcal{C}$.

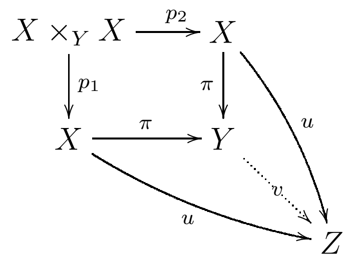
Exercise B.16. If Y' is another object representing $F \times_G h_X$, the morphism from Y' to X determined by $F \to G$ factors uniquely into $Y' \to Y \to X$, where the first morphism is an isomorphism and the second is the morphism of the definition.

Exercise B.17. The composite of two representable natural transformations is representable. If $F \to G$ is representable, then $F \times_G H \to H$ is representable for any natural transformation $H \to G$. If $F \to G$ is representable, and H is representable, then $F \times_G H$ is representable for any $H \to G$.

If F and G are contravariant functors from $\mathcal{C}$ to (Set), F is a **subfunctor** of G if, for every object S of $\mathcal{C}$, $F(S)$ is a subset of $G(S)$, and, for every morphism $u\colon T \to S$, the map $F(u)$ from $F(S)$ to $F(T)$ is the restriction of the map $G(u)$ from $G(S)$ to $G(T)$.

Let $\pi\colon X \to Y$ be a morphism in a category $\mathcal{C}$, and assume that a fibered product $X \times_Y X$ exists, with projections p_1 and p_2 from $X \times_Y X$ to X. The morphism π makes Y a **quotient** of X if it satisfies the following universal mapping property: for any morphism $u\colon X \to Z$ such that the two morphisms $u \circ p_1$ and $u \circ p_2$ from $X \times_Y X$ to Z

are equal, there is a unique morphism $v\colon Y \to Z$ such that $u = v \circ \pi$:



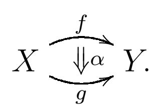
For π to make Y a quotient of X amounts to the fibered product square satisfying this dual *cocartesian* property as well as the cartesian property.

For more about categories, functors, and natural transformations, see [**65**]. For more on representable functors, see [**EGA** 0.8.1].

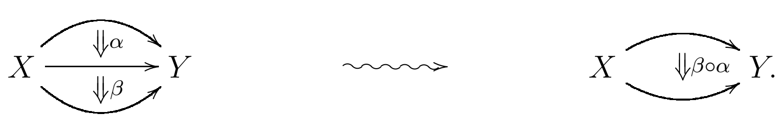
2. 2-categories

A **2-category** $\mathcal{C}$ has **objects** (denoted here X, Y, etc.), **morphisms**, sometimes called **1-morphisms** or **arrows** (denoted here f, g, etc.), and **2-morphisms** (denoted here α, β, etc.). Each morphism f has a source and target object, for which we write $f\colon X \to Y$ as before, and there are identity morphisms $1_X\colon X \to X$ for each object X, with compositions $g \circ f\colon X \to Z$ for each $f\colon X \to Y$ and $g\colon Y \to Z$. These objects and morphisms are required to satisfy the category axioms; this category is called the **underlying category** of the 2-category $\mathcal{C}$.

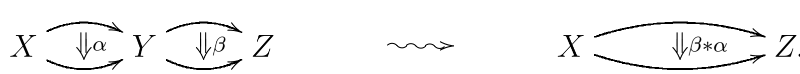
A 2-morphism α has a source morphism f and a target morphism g, with both f and g required to be morphisms with the same source and target. We write $\alpha\colon f \Rightarrow g$ to mean that α is a 2-morphism with source f and target g, and we say α is a 2-morphism **from** f **to** g. If f and g are morphisms from X to Y, this may be denoted



There are two operations on 2-morphisms. First, if $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ are 2-morphisms, with f, g, and h all morphisms with the same source and target, there is a 2-morphism, denoted $\beta \circ \alpha$, from f to h:



Second, if $\alpha\colon f \Rightarrow f'$, with f and f' morphisms from X to Y, and $\beta\colon g \Rightarrow g'$, with g and g' from Y to Z, then there is a 2-morphism $\beta * \alpha$ from $g \circ f$ to $g' \circ f'$:



As the pictures indicate, these are sometimes called **vertical** and **horizontal** composition of 2-morphisms.[2] These operations are required to satisfy the following properties, each of which is an identity between 2-morphisms:

(a) If $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$, and $\gamma\colon h \Rightarrow i$, then $(\gamma \circ \beta) \circ \alpha = \gamma \circ (\beta \circ \alpha)\colon f \Rightarrow i$.

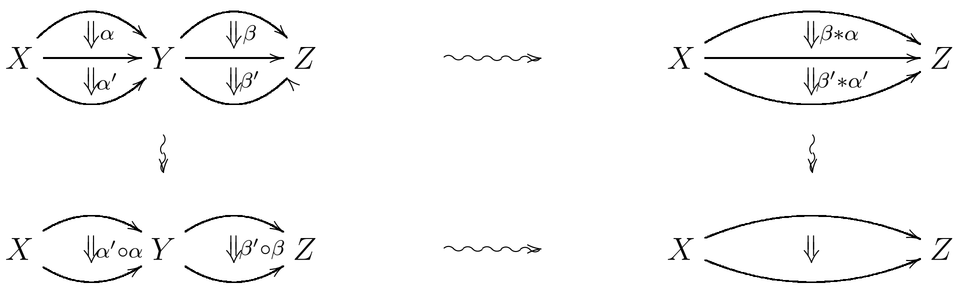
(b) For every morphism f, there is a 2-morphism $1_f\colon f \Rightarrow f$ such that $\alpha \circ 1_f = \alpha$ for all $\alpha\colon f \Rightarrow g$ and $1_f \circ \beta = \beta$ for all $\beta\colon g \Rightarrow f$. (This 1_f is unique.)

(c) For $\alpha\colon f \Rightarrow g$, with f and g from X to Y, $\alpha * 1_{1_X} = \alpha = 1_{1_Y} * \alpha$.

(d) For $f\colon X \to Y$ and $g\colon Y \to Z$, $1_g * 1_f = 1_{g \circ f}$.

(e) If $\alpha\colon f \Rightarrow f'$, with f and f' mapping X to Y, and $\beta\colon g \Rightarrow g'$, with g and g' mapping Y to Z, and $\gamma\colon h \Rightarrow h'$, with h and h' mapping Z to W, then $\gamma * (\beta * \alpha) = (\gamma * \beta) * \alpha$, as 2-morphisms from $h \circ g \circ f$ to $h' \circ g' \circ f'$.

(f) (Exchange) Given morphisms f, f', f'' from X to Y, morphisms g, g', g'' from Y to Z, and 2-morphisms $\alpha\colon f \Rightarrow f'$, $\alpha'\colon f' \Rightarrow f''$, $\beta\colon g \Rightarrow g'$, and $\beta'\colon g' \Rightarrow g''$, we have $(\beta' \circ \beta) * (\alpha' \circ \alpha) = (\beta' * \alpha') \circ (\beta * \alpha)$. In pictures:



It follows from (a) and (b) that, for any two objects X and Y, we have a category, denoted $\mathrm{HOM}(X, Y)$ (or $\mathrm{HOM}_{\mathcal{C}}(X, Y)$), whose objects are morphisms $f\colon X \to Y$, and whose arrows are 2-morphisms $\alpha\colon f \Rightarrow g$, composed by the vertical composition.

A 2-morphism $\alpha\colon f \Rightarrow g$ is a **2-isomorphism** if there is a 2-morphism $\alpha^{-1}\colon g \Rightarrow f$ with $\alpha^{-1} \circ \alpha = 1_f$ and $\alpha \circ \alpha^{-1} = 1_g$. Such α^{-1} is unique, if it exists. The notation $\alpha\colon f \overset{\sim}{\Rightarrow} g$ means that α is a 2-isomorphism. We say that morphisms f and g are **2-isomorphic** if there is a 2-isomorphism between them, and then we write $f \overset{\sim}{\Rightarrow} g$. Given any 2-category, one can throw away all 2-morphisms that are not 2-isomorphisms, with the result remaining a 2-category. (Almost all 2-morphisms appearing in this book are in fact 2-isomorphisms.)

Exercise B.18. (1) For $\alpha\colon f \Rightarrow f'$ and $\beta\colon g \Rightarrow g'$ as in the definition of $\beta * \alpha$, we have

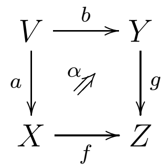
$$\beta * \alpha = (\beta * 1_{f'}) \circ (1_g * \alpha) = (1_{g'} * \alpha) \circ (\beta * 1_f).$$

In particular, the $*$-product is determined by the $\circ$-product and the $*$-product for which one of the factors in an identity 2-morphism. (2) When $\beta * \alpha$ is defined, if α and β are 2-isomorphisms, then $\beta * \alpha$ is a 2-isomorphism, with inverse $\beta^{-1} * \alpha^{-1}$.

[2] The reader should be warned that the symbols $\circ$, $*$, $\bullet$, $\cdot$, as well as juxtaposition, and probably others, have been used for one or the other of these operations.

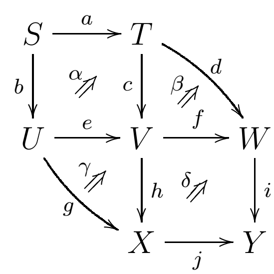
The 2-morphisms in a 2-category are sometimes called *2-cells*. In this case the morphisms are called *1-cells*, and the objects may be called *0-cells*.

A diagram



means that a 2-morphism $\alpha\colon f\,a \Rightarrow g\,b$ is specified. (If the arrow is pointed in the other direction, it indicates a 2-morphism from $g\,b$ to $f\,a$.) We say that the diagram **2-commutes** when a 2-isomorphism $\alpha\colon f\,a \overset{\sim}{\Rightarrow} g\,b$ is given. When $f\,a = g\,b$, the diagram is said to **strictly commute**, and α is taken to be $1_{f\,a} = 1_{g\,b}$; in this case the arrow $\Rightarrow$ in the diagram may be replaced by an equality sign $=$. The same terminology is used when the square is replaced by any polygon, with arrows starting at some vertex and moving in opposite directions toward another vertex.

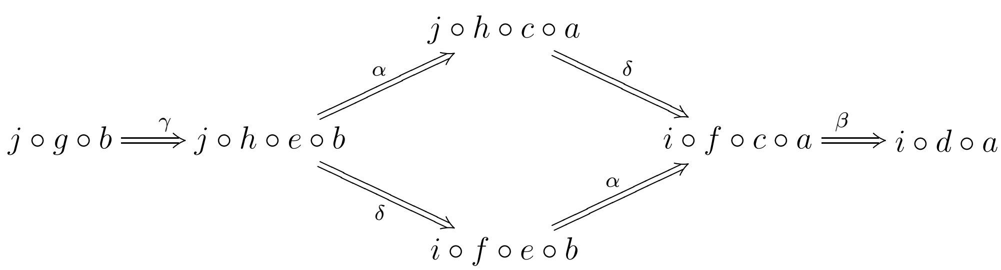
The axioms, particularly the exchange property, allow one to compose 2-morphisms across diagrams, with the result being independent of choices. For example, given a diagram



one gets a 2-morphism from $j\,g\,b$ to $i\,d\,a$, by first doing γ, then α and δ (in either order), and finally doing β. Officially, this 2-morphism is

$$(1_i * \beta * 1_a) \circ (\delta * \alpha) \circ (1_j * \gamma * 1_b),$$

noting that $\delta * \alpha = (\delta * 1_{c\,a}) \circ (1_{j\,h} * \alpha) = (1_{i\,f} * \alpha) \circ (\delta * 1_{e\,b})$. Sometimes one can express an equality among 2-morphisms by saying that the results of such pastings of polygons around the sides of a solid polytope in 3-space are the same, but these diagrams (with their labels) are not easy to draw, nor are they easy to manipulate to prove identities. In fact, it is often useful to express an equality among 2-morphisms by an ordinary commutative diagram involving 2-morphisms in a HOM-category. For example, the above situation can be expressed by the diagram



in the category HOM(S, Y), with the central square commuting. When no confusion is possible, we omit the identity 2-isomorphisms from the labels over double arrows; in this example, the γ over the first double arrow is short for $1_j * \gamma * 1_b$, and similarly for the others. Similarly, (1) of Exercise B.18 says that the diagrams

$$\begin{array}{ccc} g \circ f & \stackrel{\alpha}{\Longrightarrow} & g \circ f' \\ & \underset{\beta*\alpha}{\searrow\!\!\searrow} & \Downarrow \beta \\ & & g' \circ f' \end{array} \qquad\qquad \begin{array}{ccc} g \circ f & & \\ \beta \Downarrow & \overset{\beta*\alpha}{\searrow\!\!\searrow} & \\ g' \circ f & \underset{\alpha}{\Longrightarrow} & g' \circ f' \end{array}$$

commute.

Exercise B.19. Let $h\colon X \to X$ be a morphism in a 2-category, and let $\theta\colon 1_X \stackrel{\sim}{\Rightarrow} h$ be a 2-isomorphism. Show that $\theta * 1_h = 1_h * \theta$ from h to $h \circ h$, i.e., the diagram

$$\begin{array}{ccc} h & = = = & 1_X \circ h \\ \| & & \Downarrow \theta \\ h \circ 1_X & \underset{\theta}{\Longrightarrow} & h \circ h \end{array}$$

commutes in the category HOM(X, X).

Exercise B.20. Properties (d) and (f) say that the assignment

$$\mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Z) \to \mathrm{HOM}(X,Z)$$

that takes (f, g) to $g \circ f$, and (α, β) to $\beta * \alpha$, is a functor. Property (e) implies that, for X, Y, Z and W, the diagram of categories

$$\begin{array}{ccc} \mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,Z) \times \mathrm{HOM}(Z,W) & \longrightarrow & \mathrm{HOM}(X,Z) \times \mathrm{HOM}(Z,W) \\ \downarrow & & \downarrow \\ \mathrm{HOM}(X,Y) \times \mathrm{HOM}(Y,W) & \longrightarrow & \mathrm{HOM}(X,W) \end{array}$$

commutes. Property (c) implies that the composite functor

$$\mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,X) \times \mathrm{HOM}(X,Y) \to \mathrm{HOM}(X,Y),$$

where the first takes f to $(1_X, f)$ and α to $(1_{1_X}, \alpha)$, is the identity functor.

We give several examples, starting with the prototype from geometry.

Example B.5. There is a 2-category (Top), whose objects are topological spaces, whose morphisms are continuous maps, and whose 2-morphisms come from homotopies — but here we must take appropriate equivalence classes. Given continuous maps f and g from X to Y, a **homotopy** from f to g is a continuous mapping

$$H\colon X \times [0,1] \to Y$$

with $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Call two homotopies H_0 and H_1 **equivalent** if there is a continuous mapping

$$K\colon X \times [0,1] \times [0,1] \to Y$$

with $K(x,t,0) = H_0(x,t)$, $K(x,t,1) = H_1(x,t)$, $K(x,0,u) = f(x)$, and $K(x,1,u) = g(x)$ for all $x \in X$, $t, u \in [0,1]$. (This is an equivalence relation.) A **2-morphism** from f to g is defined to be an equivalence class of homotopies from f to g.

If f, g, and h map X to Y, and H_1 is a homotopy from f to g, and H_2 a homotopy from g to h, define $H_2 \circ H_1$ by

$$H_2 \circ H_1(x,t) = \begin{cases} H_1(x,2t), & 0 \le t \le 1/2 \\ H_2(x,2t-1), & 1/2 \le t \le 1 \end{cases}.$$

This passes to equivalence of homotopies, so defines the vertical composition $\beta \circ \alpha$ of 2-morphisms. It is associative by the same calculation made to show the associativity of fundamental groups.

If f and f' map X to Y, and g and g' map Y to Z, and H_1 is a homotpy from f to f', and H_2 is a homotopy from g to g', define a homotopy $H_2 * H_1$ from $g \circ f$ to $g' \circ f'$ by

$$(H_2 * H_1)(x,t) = H_2(H_1(x,t),t) \qquad x \in X, \quad 0 \le t \le 1.$$

This passes to equivalence classes, defining the horizontal product $\beta * \alpha$ of 2-morphisms.

Exercise B.21. Verify that these operations make (Top) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

Example B.6. There is a 2-category (CC) of chain complexes of abelian groups (and similarly, a 2-category (CC_R) of chain complexes of R-modules, for a commutative ring R). The objects are the usual chain complexes $C = C_\bullet$, with boundary homomorphisms $d_n \colon C_n \to C_{n-1}$ satisfying $d_{n-1} \circ d_n = 0$. A morphism $f = f_\bullet$ from C to D is a collection of homomorphisms $f_n \colon C_n \to D_n$, commuting with the boundary maps. A chain homotopy $\alpha = \alpha_\bullet$ from f to g is a collection of homomorphisms $\alpha_n \colon C_n \to D_{n+1}$ such that $d_{n+1} \circ \alpha_n + \alpha_{n-1} \circ d_n = g_n - f_n$ for all n. Call two chain homomotopies α and β from f to g **equivalent** if there is a collection $\theta = \theta_\bullet$ of homomorphisms $\theta_n \colon C_n \to D_{n+2}$ such that

$$d_{n+2} \circ \theta_n - \theta_{n-1} \circ d_n = \beta_n - \alpha_n$$

for all n. (This is an equivalence relation.) A **2-morphism** from f to g is an equivalence class of such chain homotopies.

If f, g, and h map C to D, α is a chain homotopy from f to g, and β is a chain homotopy from g to h, define the chain homotopy $\beta \circ \alpha$ from f to h by the formula $(\beta \circ \alpha)_n = \alpha_n + \beta_n$. This passes to equivalence classes, so defines a vertical composition of 2-morphisms. If f and f' map C to D, and g and g' map D to E, and α is a chain homotopy from f to f' and β is a chain homotopy from g to g', define the chain homotopy $\beta * \alpha$ from $g \circ f$ to $g' \circ f'$ by the formula

$$(\beta * \alpha)_n = g_{n+1} \circ \alpha_n + \beta_n \circ f'_n.$$

(This is equivalent to the alternative $\beta_n \circ f_n + g'_{n+1} \circ \alpha_n$.) This respects the equivalence, so defines a horizontal composition of 2-morphisms.

Exercise B.22. Verify that these objects, morphisms, and 2-morphisms satisfy the axioms to form a 2-category.

Example B.7. The 2-category (Grp) has groups as objects, group homomorphisms as morphisms, and, if f and g are homomorphisms from X to Y, a 2-morphism from f to g is an element y in Y such that

$$g(x) = y^{-1} \cdot f(x) \cdot y \quad \text{for all } x \in X.$$

If z gives a 2-morphism from g to h, the composition $z \circ y$ from f to h is given by $y \cdot z$. If $y\colon f \Rightarrow f'$, with f and f' from X to Y, and $z\colon g \Rightarrow g'$, with g and g' from Y to Z, then $z * y\colon g \circ f \Rightarrow g' \circ f'$ is given by the element $g(y) \cdot z = z \cdot g'(y)$ of Z.

Exercise B.23. Verify that these operations make (Grp) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

The following example, with variations, is the key example for this text.

Example B.8. Categories form a 2-category (Cat). Its objects are categories $\mathcal{C}$, its morphisms are functors $F\colon \mathcal{C} \to \mathcal{D}$, and its 2-morphisms $\alpha\colon F \Rightarrow G$ are natural transformations from F to G. If $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ are natural transformations between functors from $\mathcal{C}$ to $\mathcal{D}$, then $\beta \circ \alpha\colon F \Rightarrow H$ is the natural transformation that takes an object X of $\mathcal{C}$ to the morphism $\beta_X \circ \alpha_X$ from $F(X)$ to $H(X)$. If F, F' are functors from $\mathcal{C}$ to $\mathcal{D}$, with $\alpha\colon F \Rightarrow F'$, and G, G' are functors from $\mathcal{D}$ to $\mathcal{E}$, with $\beta\colon G \Rightarrow G'$, define $\beta * \alpha\colon G \circ F \Rightarrow G' \circ F'$ to take the object X of $\mathcal{C}$ to the morphism

$$G'(\alpha_X) \circ \beta_{F(X)} = \beta_{F'(X)} \circ G(\alpha_X)$$

of $\mathcal{E}$.

Exercise B.24. Verify that these operations make (Cat) into a 2-category. (Not all 2-morphisms are 2-isomorphisms.)

Thinking of groupoids of sets as categories shows that groupoids of sets form a 2-category (Gpd). More generally:

Example B.9. Let $\mathcal{S}$ be a category. Let ($\mathcal{S}$-Gpd) be the category whose objects are $\mathcal{S}$-groupoids, whose morphisms are morphisms of $\mathcal{S}$-groupoids (see Chapter 3). If (φ, Φ) and (ψ, Ψ) are morphisms from $R' \rightrightarrows U'$ to $R \rightrightarrows U$, define a 2-morphism from (φ, Φ) to (ψ, Ψ) to be a morphism $\alpha\colon U' \to R$ in $\mathcal{S}$ such that $s \circ \alpha = \varphi$, $t \circ \alpha = \psi$, and the diagram

$$\begin{array}{ccc} R' & \xrightarrow{(\alpha s', \Psi)} & R \;{}_t\!\times_s R \\ {\scriptstyle (\Phi, \alpha t')}\downarrow & & \downarrow{\scriptstyle m} \\ R \;{}_t\!\times_s R & \xrightarrow{m} & R \end{array}$$

commutes.

Exercise B.25. Make ($\mathcal{S}$-Gpd) into a 2-category, in which all 2-morphisms are 2-isomorphisms.

Exercise B.26. There is a category whose objects are sets, with an arrow from X to Y being a subset f of $X \times Y$. Define the composition of f with g from Y to Z to be the set of (x, z) in $X \times Z$ such that there is a y in Y with (x, y) in f and (y, z) in g.

This category can be enriched to a 2-category by defining a unique 2-cell from subsets f and g of $X \times Y$ if f is contained in g, with no 2-cell from f to g otherwise. Verify that this is a 2-category.

Exercise B.27. If $\mathcal{C}$ is a 2-category, construct a category $\overline{\mathcal{C}}$, whose objects are the same as the objects of $\mathcal{C}$, but whose morphisms from X to Y are equivalence classes of morphisms $f\colon X \to Y$ in $\mathcal{C}$, where f is equivalent to g if there is a 2-isomorphism from f to g. Show that this is an equivalence relation, and that $\overline{\mathcal{C}}$ is a category, with the canonical map from the underlying category of $\mathcal{C}$ to $\overline{\mathcal{C}}$ being a functor. Examples are: the category of topological spaces with homotopy classes of mappings; the category of groups with homomorphisms up to inner automorphism; the category of categories, with functors up to natural isomorphism. This category $\overline{\mathcal{C}}$ is sometimes called the *classifying category* of $\mathcal{C}$, see [**11**].

Exercise B.28. Any category $\mathcal{C}$ determines a 2-category, with the same objects and morphisms, and with the only 2-morphisms being identities 1_f, for morphisms f in $\mathcal{C}$.

We say that a 2-category **is a 1-category** if its only 2-morphisms are identities. In this spirit, one says that a category **is a 0-category** if its only morphisms are identity maps.

Exercise B.29. If $\mathcal{C}$ is a 2-category, a category $\mathcal{C}'$ can be constructed as follows. The objects of $\mathcal{C}'$ are the objects of $\mathcal{C}$; the morphisms of $\mathcal{C}'$ from X to Y are the 2-morphisms $\alpha\colon f \Rightarrow g$, where f and g are maps from X to Y in $\mathcal{C}$. The composite of α followed by β is $\beta * \alpha$. Verify that $\mathcal{C}'$ is a category.

Exercise B.30. (1) If $\mathcal{C}$ is a 2-category, and $f\colon X \to Y$ a morphism in $\mathcal{C}$, we have, for every object S of $\mathcal{C}$, a functor

$$f^S\colon \operatorname{HOM}(S,X) \to \operatorname{HOM}(S,Y)$$

taking $h\colon S \to X$ to $f \circ h\colon S \to Y$, and $\alpha\colon h \Rightarrow h'$ to $1_f * \alpha\colon f \circ h \Rightarrow f \circ h'$. Similarly, there are functors

$$f_S\colon \operatorname{HOM}(Y,S) \to \operatorname{HOM}(X,S)$$

taking $h\colon Y \to S$ to $h \circ f\colon X \to S$ and $\alpha\colon h \Rightarrow h'$ to $\alpha * 1_f\colon h \circ f \Rightarrow h' \circ f$.

(2) If also $g\colon Y \to Z$, then $(g \circ f)^S = g^S \circ f^S$ and $(g \circ f)_S = f_S \circ g_S$. If $f = 1_X$, then $f^S = 1_{\operatorname{HOM}(S,X)}$ and $f_S = 1_{\operatorname{HOM}(X,S)}$. It follows that, if f is an isomorphism, then each functor f^S and f_S is an isomorphism of categories.

(3) If f and g are morphisms from X to Y, and $\sigma\colon f \Rightarrow g$ is a 2-morphism, then σ determines a natural transformation σ^S from f^S to g^S (taking $h\colon S \to X$ to $\sigma * 1_h$), and a natural transformation σ_S from f_S to g_S (taking $h\colon Y \to S$ to $1_h * \sigma$). If also $\tau\colon g \Rightarrow h$, then $(\tau \circ \sigma)^S = \tau^S \circ \sigma^S$. If $\sigma = 1_f$, then σ^S is the identity natural isomorphism on f^S. Hence, if σ is invertible, then σ^S is a natural isomorphism.

(4) For fixed objects X, Y, and S of $\mathcal{C}$, there is a functor $\operatorname{HOM}(X,Y) \to \operatorname{HOM}_{(\mathrm{Cat})}(\operatorname{HOM}(S,X), \operatorname{HOM}(S,Y))$ taking f to f^S and σ to σ^S.

Just as two topological spaces can be homotopy equivalent, there is a notion for two objects in any 2-category to be 2-isomorphic. In fact, there are several ways to say this:

PROPOSITION B.10. *Let $f\colon X \to Y$ be a morphism in a 2-category $\mathcal{C}$. The following are equivalent:*

(1) *There is a morphism $g\colon Y \to X$ together with 2-isomorphisms $\phi\colon 1_X \overset{\sim}{\Rightarrow} g \circ f$ and $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$.*

(2) *There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\phi\colon 1_X \overset{\sim}{\Rightarrow} g \circ f$ and $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$ such that $1_f * \phi = \psi * 1_f$ (as 2-isomorphisms from f to fgf) and $\phi * 1_g = 1_g * \psi$ (as 2-isomorphisms from g to gfg). That is, the diagrams*

$$\begin{array}{ccc} f & = & 1_Y \circ f \\ \Vert & & \Downarrow \psi \\ f \circ 1_X & \underset{\phi}{\Longrightarrow} & f \circ g \circ f \end{array} \qquad\qquad \begin{array}{ccc} g & = & 1_X \circ g \\ \Vert & & \Downarrow \phi \\ g \circ 1_Y & \underset{\psi}{\Longrightarrow} & g \circ f \circ g \end{array}$$

commute, in the categories $\mathrm{HOM}(X,Y)$ *and* $\mathrm{HOM}(Y,X)$ *respectively.*

(3) *For every object S of $\mathcal{C}$, the functor $f^S\colon \mathrm{HOM}(S,X) \to \mathrm{HOM}(S,Y)$ is an equivalence of categories.*

(4) *The functors f^X and f^Y are equivalences of categories.*

PROOF. We show first how (1) implies (3). By Exercise B.30, we have the functor $g^S\colon \mathrm{HOM}(S,Y) \to \mathrm{HOM}(S,X)$, and we have a natural isomorphism $\phi^S\colon 1_{\mathrm{HOM}(S,X)} \overset{\sim}{\Rightarrow} g^S \circ f^S$. Similarly, we have a natural isomorphism $\psi^S\colon 1_{\mathrm{HOM}(S,Y)} \overset{\sim}{\Rightarrow} f^S \circ g^S$.

Next we prove that (4) implies (2), which finishes the proof since (4) is a special case of (3) and (1) is a special case of (2). Since f^Y is essentially surjective, there is a morphism $g\colon Y \to X$ and a 2-isomorphism $\psi\colon 1_Y \overset{\sim}{\Rightarrow} f \circ g$. Since f^X is full and faithful, there is a unique 2-morphism $\phi\colon 1_X \Rightarrow g \circ f$ such that $f^X(\phi)$ is the 2-isomorphism $\psi * 1_f$ from $f = 1_Y \circ f$ to $f \circ g \circ f$; this ϕ is an isomorphism since $f^X(\phi)$ is an isomorphism (Exercise B.6). Since $f^X(\phi) = 1_f * \phi$, we have one of the required equations $1_f * \phi = \psi * 1_f$. To prove that the 2-morphisms $\phi * 1_g$ and $1_g * \psi$ from g to $g \circ f \circ g$ are equal in $\mathrm{HOM}(Y,X)$, it suffices to show that their images by the faithful functor f^Y are equal, i.e., to show that $1_f * (\phi * 1_g) = 1_f * (1_g * \psi)$. Now

$$\begin{aligned} 1_f * (\phi * 1_g) = (1_f * \phi) * 1_g = (\psi * 1_f) * 1_g = \psi * 1_{fg} \\ = 1_{fg} * \psi = (1_f * 1_g) * \psi = 1_f * (1_g * \psi), \end{aligned}$$

as required; the fourth equality used Exercise B.19. □

This proof shows that, given f, g, ϕ, and ψ, either of the equations $1_f * \phi = \psi * 1_f$ or $\phi * 1_g = 1_g * \psi$ implies the other.

DEFINITION B.11. We call a morphism $f\colon X \to Y$ in a 2-category **2-invertible** or a **2-equivalence**, if it satisfies the conditions of the proposition. (We do not use the more natural term *2-isomorphism*, to avoid confusion with invertible 2-morphisms.)

On the other hand, if there exists a 2-invertible morphism $f\colon X \to Y$, then we call the objects X and Y **2-isomorphic**, as there is no danger of confusion in this context.

A triple (g, ϕ, η) satisfying the conditions of (2) may be called a **2-inverse** of f. A quadruple satisfying the conditions of (2) is sometimes called an **adjoint equivalence**. In practice, one uses (1) to check that a morphism is 2-invertible, but one uses the full data of (2) in making constructions.

EXERCISE B.31. Show that the conditions of the proposition are equivalent to each of the following:

(5) For every object S of $\mathcal{C}$, the functor $f_S\colon \mathrm{HOM}(Y, S) \to \mathrm{HOM}(X, S)$ is an equivalence of categories.
(6) The functors f_X and f_Y are equivalences of categories.
(7) The functors f_X and f^Y are essentially surjective.
(8) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\phi\colon 1_X \stackrel{\sim}{\Rightarrow} g \circ f$ and $\eta\colon f \circ g \stackrel{\sim}{\Rightarrow} 1_Y$ such that the composition $f = f \circ 1_X \stackrel{\phi}{\Rightarrow} f \circ g \circ f \stackrel{\eta}{\Rightarrow} 1_Y \circ f = f$ is equal to 1_f, and the composition $g = 1_X \circ g \stackrel{\phi}{\Rightarrow} g \circ f \circ g \stackrel{\eta}{\Rightarrow} g \circ 1_Y = g$ is equal to 1_g.
(9) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\psi\colon 1_Y \stackrel{\sim}{\Rightarrow} f \circ g$ and $\theta\colon g \circ f \stackrel{\sim}{\Rightarrow} 1_X$ such that the composition $f = 1_Y \circ f \stackrel{\psi}{\Rightarrow} f \circ g \circ f \stackrel{\theta}{\Rightarrow} f \circ 1_X = f$ is equal to 1_f, and the composition $g = g \circ 1_Y \stackrel{\psi}{\Rightarrow} g \circ f \circ g \stackrel{\theta}{\Rightarrow} 1_X \circ g = g$ is equal to 1_g.
(10) There is a morphism $g\colon Y \to X$ and 2-isomorphisms $\theta\colon g \circ f \stackrel{\sim}{\Rightarrow} 1_X$ and $\eta\colon f \circ g \stackrel{\sim}{\Rightarrow} 1_Y$ such that the diagrams

$$\begin{array}{ccc} f \circ g \circ f & \stackrel{\theta}{\Longrightarrow} & f \circ 1_X \\ {\scriptstyle \eta}\Downarrow & & \| \\ 1_Y \circ f & = & f \end{array} \qquad\qquad \begin{array}{ccc} g \circ f \circ g & \stackrel{\theta}{\Longrightarrow} & 1_X \circ g \\ {\scriptstyle \eta}\Downarrow & & \| \\ g \circ 1_Y & = & g \end{array}$$

commute.

It follows from Proposition B.20 in the next section, together with (9) of the preceding exercise, that if (g, ϕ, ψ) is a 2-inverse of f, then any other 2-inverse of f has the form (g', ϕ', ψ'), for a unique 2-isomorphism $\theta\colon g \stackrel{\sim}{\Rightarrow} g'$ with $\phi' = (\theta * 1_f) \circ \phi$ and $\psi' = (1_f * \theta) \circ \psi$.

EXERCISE B.32. In the 2-category (Top), two spaces are 2-isomorphic exactly when they have the same homotopy type. In the 2-category (Grp), two groups are 2-isomorphic if and only if they are isomorphic groups. In the 2-category (Cat), two categories are 2-isomorphic when they are equivalent.

EXERCISE B.33. Show that the condition of being 2-isomorphic is an equivalence relation on the objects of a 2-category.

When applied to the 2-category (Cat), Proposition B.10 and Exercise B.31 give a variety of criteria for a functor F from a category $\mathcal{C}$ to a category $\mathcal{D}$ to be an equivalence of categories. Note that the equivalence with (9) recovers the result of Exercise B.8. For the 2-category (Top), one recovers a criterion of Vogt [**90**]. The general statement, in the form that (1) implies (9), appears in [**63**], where it is attributed to a combination of folklore and R. Street.

Definition B.12. A **sub-2-category** $\mathcal{C}'$ of a 2-category $\mathcal{C}$ is obtained by selecting some of the objects, some of the morphisms, and some of the 2-morphisms, of $\mathcal{C}$, in such a way that all identities 1_X of selected objects and 1_f of selected morphisms are selected, and all composites $g \circ f$ and $\beta \circ \alpha$ of selected morphisms or 2-morphisms are selected, as is the product $\beta * \alpha$, whenever such composites or products are defined in $\mathcal{C}$. It is easy to verify that $\mathcal{C}'$ is a 2-category. A sub-2-category $\mathcal{C}'$ is called a **full sub-2-category** of $\mathcal{C}$ if any morphism in $\mathcal{C}$ between two objects of $\mathcal{C}'$ is in $\mathcal{C}'$, and any 2-morphism in $\mathcal{C}$ between two morphisms in $\mathcal{C}'$ is in $\mathcal{C}'$.

Example B.13. The 2-category (Grp) of groups forms a full sub-2-category of the 2-category (Gpd) of groupoids of sets, which in turn forms a full sub-2-category of the 2-category (Cat) of categories.

Most "mappings" from one 2-category to another will not preserve all the structure strictly; rather, the expected identities will be true only up to specified 2-isomorphisms. These "pseudofunctors" will be studied in Section 4. We include here a brief discussion of the stronger notion, called a 2-functor, as a warmup. A **2-functor** $F\colon \mathcal{C} \to \mathcal{D}$ from one 2-category to another assigns to each object X in $\mathcal{C}$ an object $F(X)$ in $\mathcal{D}$, to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$, and to each 2-morphism $\alpha\colon f \Rightarrow g$ in $\mathcal{C}$ a 2-morphism $F(\alpha)\colon F(f) \Rightarrow F(g)$ in $\mathcal{D}$, satisfying:

(a) $F(1_X) = 1_{F(X)}$ for all objects X of $\mathcal{C}$;
(b) $F(1_f) = 1_{F(f)}$ for all morphisms f of $\mathcal{C}$;
(c) $F(g \circ f) = F(g) \circ F(f)$ for $f\colon X \to Y$, $g\colon Y \to Z$ in $\mathcal{C}$;
(d) $F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$ for $\alpha\colon f \Rightarrow g$, $\beta\colon g \Rightarrow h$ in $\mathcal{C}$;
(e) $F(\beta * \alpha) = F(\beta) * F(\alpha)$ when $\beta * \alpha$ is defined in $\mathcal{C}$.

This gives a functor between the underlying categories (called the **underlying functor**). For objects X and Y of $\mathcal{C}$, it also gives a functor $\mathrm{HOM}(X, Y) \to \mathrm{HOM}(F(X), (F(Y))$ (by (b) and (d)). For example, the inclusion of a sub-2-category in a 2-category is a 2-functor.

Exercise B.34. Construct a 2-functor from the 2-category (Top) of topological spaces to the 2-category (Gpd) of groupoids, that takes a space X to its fundamental groupoid.

Example B.14. There is a 2-functor from the 2-category (Top) to the 2-category (CC) of chain complexes. This takes a topological space X to the chain complex $C_\bullet(X)$ of nondegenerate cubical chains.[3] A continuous mapping $f\colon X \to Y$ is sent to the chain mapping $f_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ that takes σ to $f \circ \sigma$. A homotopy $H\colon X \times [0,1] \to Y$

[3] $C_n(X)$ is the free module on the set of continuous maps $\sigma\colon [0,1]^n \to X$, modulo the submodule generated by those σ such that, for some $1 \le i \le n$, $\sigma(t_1, \ldots, t_n)$ is a constant function of t_i. The boundary $d_n\colon C_n(X) \to C_{n-1}(X)$ is defined by the formula $d_n = \sum_{i=1}^n (-1)^i (\partial_i^0 - \partial_i^1)$, where $\partial_i^\epsilon(\sigma)(t_1, \ldots, t_{n-1}) = \sigma(t_1, \ldots, t_{i-1}, \epsilon, t_i, \ldots, t_{n-1})$.

from f to g determines a chain homotopy α_H from $f_\bullet$ to $g_\bullet$, by the formula

$$\alpha_H(\sigma)(t_1, \ldots, t_{n+1}) = H(\sigma(t_2, \ldots, t_{n+1}), t_1).$$[4]

Exercise B.35. Verify that α_H is a chain homotopy. Show that equivalent homotopies from f to g determine equivalent chain homotopies from $f_\bullet$ to $g_\bullet$, so a 2-morphism in (Top) determines a 2-morphism in (CC). Show that taking X to $C_\bullet(X)$, f to $f_\bullet$, and an equivalence class of H's to the equivalence class of α_H's, determines a 2-functor from (Top) to (CC).

If F and G are 2-functors from a 2-category $\mathcal{C}$ to a 2-category $\mathcal{D}$, a **2-natural transformation** θ from F to G assigns to each object X of $\mathcal{C}$ a morphism $\theta_X \colon F(X) \to G(X)$ in $\mathcal{D}$, satisfying two properties. First, for all $f \colon X \to Y$ in $\mathcal{C}$, the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle\theta_X}\downarrow & & \downarrow{\scriptstyle\theta_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

must commute. This says that θ is a natural transformation between the underlying functors on the underlying categories. The second property says that for any $f, g \colon X \to Y$ and 2-morphism $\alpha \colon f \Rightarrow g$ in $\mathcal{C}$, the two morphisms $1_{\theta_Y} * F(\alpha)$ and $G(\alpha) * 1_{\theta_X}$, pictured by

from $\theta_Y \circ F(f) = G(f) \circ \theta_X$ to $\theta_Y \circ F(g) = G(g) \circ \theta_X$ must be equal. A 2-natural transformation is a **2-natural isomorphism** if each θ_X is an isomorphism.

Exercise B.36. Define vertical and horizontal composition of 2-natural transformations, by the formulas: $(\beta \circ \alpha)_X = \beta_X \circ \alpha_X$; and $(\beta * \alpha)_X = G'(\alpha_X) \circ \beta_{F(X)} = \beta_{F'(X)} \circ G(\alpha_X)$, the latter when α (resp. β) is a 2-natural transformation from F to F' (resp. G to G'). Show that, with these operations, 2-categories, 2-functors, and 2-natural transformations form the objects, arrows, and 2-cells of a 2-category (2-Cat).

One can call a 2-functor $F \colon \mathcal{C} \to \mathcal{D}$ a **strict 2-isomorphism** if there is a 2-functor $G \colon \mathcal{D} \to \mathcal{C}$ with $G \circ F = 1_{\mathcal{C}}$ and $F \circ G = 1_{\mathcal{D}}$. This notion is much too strong to be useful. Somewhat better is the following: A 2-functor $F \colon \mathcal{C} \to \mathcal{D}$ between 2-categories is a **2-equivalence** if there is a 2-functor $G \colon \mathcal{D} \to \mathcal{C}$ and there are 2-natural isomorphisms from $G \circ F$ to $1_{\mathcal{C}}$ and from $F \circ G$ to $1_{\mathcal{D}}$.

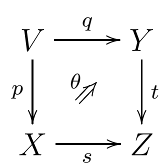
Exercise B.37. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a 2-functor. The following are equivalent: (1) F is a 2-equivalence. (2) F determines an equivalence between the underlying categories, and, for all objects X and Y of $\mathcal{C}$, the induced functor $\mathrm{HOM}(X, Y) \to \mathrm{HOM}(F(X), F(Y))$ is a *strict* isomorphism of categories.

[4]Readers who prefer simplices may use the method of acyclic models to obtain a similar 2-functor involving simplicial complexes.

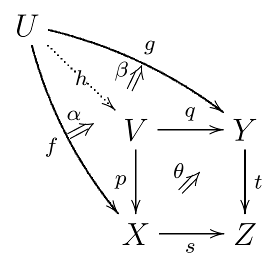
These 2-functors are relatively rare in the world of 2-categories, and the notion of "isomorphism" that appears in the preceding exercise is too strong to be very useful; a more flexible notation is discussed in Section 4.

Definition B.15. The **opposite** 2-category $\mathcal{C}^{\mathrm{op}}$ of a 2-category $\mathcal{C}$ is obtained by reversing the direction of the 1- morphisms, keeping the direction of the 2-morphisms the same. Thus if f and g are morphisms from X to Y in $\mathcal{C}$, and α is a 2-morphism from f to g, then in $\mathcal{C}^{\mathrm{op}}$ there are morphisms f and g from Y to X, with α a 2-morphism from f to g.

Definition B.16. A 2-commutative diagram

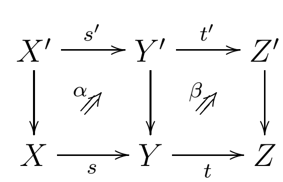


(with θ a 2-isomorphism from $s\,p$ to $t\,q$) is said to be **2-cartesian** if it satisfies the following universal property: For any morphisms $f\colon U \to X$ and $g\colon U \to Y$ and 2-isomorphism $\phi\colon sf \overset{\sim}{\Rightarrow} tg$, there is a morphism $h\colon U \to V$ and 2-isomorphisms $\alpha\colon f \overset{\sim}{\Rightarrow} ph$ and $\beta\colon qh \overset{\sim}{\Rightarrow} g$ such that $\phi = (1_t * \beta) \circ (\theta * 1_h) \circ (1_s * \alpha)$:



In addition, we must have the following uniqueness: if $h'\colon U \to V$ and $\alpha'\colon f \overset{\sim}{\Rightarrow} ph'$ and $\beta'\colon qh' \overset{\sim}{\Rightarrow} g$ also have $\phi = (1_t * \beta') \circ (\theta * 1_{h'}) \circ (1_s * \alpha')$, then there is a *unique* 2-isomorphism $\rho\colon h \overset{\sim}{\Rightarrow} h'$ such that $\alpha' = (1_p * \rho) \circ \alpha$ and $\beta = \beta' \circ (1_q * \rho)$. In this case we will call V a **fibered product** of X and Y over Z, and write $V = X \times_Z Y$, but note that the morphisms, and especially the 2-isomorphism, are understood to be part of the structure.

Exercise B.38. Given a diagram



if the two squares are 2-cartesian, show that the resulting diagram

$$\begin{array}{ccc} X' & \xrightarrow{t's'} & Z' \\ \downarrow & \overset{\gamma}{\nearrow} & \downarrow \\ Y & \xrightarrow[ts]{} & Z \end{array}$$

is also 2-cartesian, with $\gamma = (\beta * 1_{s'}) \circ (1_t * \alpha)$. State and prove analogues of the other parts of Exercises B.11 and B.12 for 2-cartesian diagrams.

Definition B.17. $^{(*)}$ The notion of a quotient in a 2-category is more complicated than that in an ordinary category. To define it, we need some notation for some fibered products. Let $\pi \colon X \to Y$ be a morphism in a 2-category $\mathcal{C}$, and assume there is a fibered product $X_1 = X \times_Y X$, with its projections p_1 and p_2 from X_1 to X and 2-isomorphism $\theta \colon \pi \circ p_1 \Rightarrow \pi \circ p_2$. In addition, assume that there is a fibered product $X_2 = X \times_Y X \times_Y X$, with its projections $q_1, q_2, q_3 \colon X_2 \to X$, with associated 2-isomorphisms

$$\theta_{12} \colon \pi \circ q_1 \Rightarrow \pi \circ q_2, \quad \theta_{23} \colon \pi \circ q_2 \Rightarrow \pi \circ q_3.$$

Set $\theta_{13} = \theta_{23} \circ \theta_{12} \colon \pi \circ q_1 \Rightarrow \pi \circ q_3$. For $1 \le i < j \le 3$ we have projections $p_{ij} \colon X_2 \to X_1$, with 2-isomorphisms $\alpha_{ij} \colon q_i \Rightarrow p_1 \circ p_{ij}$ and $\alpha_{ji} \colon q_j \Rightarrow p_2 \circ p_{ij}$, such that the diagrams

$$\begin{array}{ccc} \pi \circ q_i & \overset{\alpha_{ij}}{\Longrightarrow} & \pi \circ p_1 \circ p_{ij} \\ {\scriptstyle\theta_{ij}}\Downarrow & & \Downarrow{\scriptstyle\theta} \\ \pi \circ q_j & \underset{\alpha_{ji}}{\Longrightarrow} & \pi \circ p_2 \circ p_{ij} \end{array}$$

commute. Define $\alpha_1 = \alpha_{13} \circ \alpha_{12}^{-1} \colon \pi \circ p_1 \circ p_{12} \Rightarrow \pi \circ p_1 \circ p_{13}$, $\alpha_2 = \alpha_{23} \circ \alpha_{21}^{-1} \colon \pi \circ p_2 \circ p_{12} \Rightarrow \pi \circ p_1 \circ p_{23}$, and $\alpha_3 = \alpha_{32} \circ \alpha_{31}^{-1} \colon \pi \circ p_2 \circ p_{13} \Rightarrow \pi \circ p_2 \circ p_{23}$.[5]

We say that $\pi \colon X \to Y$ makes Y a **2-quotient** of X if it satisfies the following universal mapping property. For any morphism $u \colon X \to Z$, and any 2-isomorphism $\tau \colon u \circ p_1 \Rightarrow u \colon p_2$, such that the diagram

$$\begin{array}{ccccc} u \circ p_1 \circ p_{12} & \overset{\alpha_1}{\Longrightarrow} & u \circ p_1 \circ p_{13} & \overset{\tau}{\Longrightarrow} & u \circ p_2 \circ p_{13} \\ {\scriptstyle\tau}\Downarrow & & & & \Downarrow{\scriptstyle\alpha_3} \\ u \circ p_2 \circ p_{12} & \underset{\alpha_2}{\Longrightarrow} & u \circ p_1 \circ p_{23} & \underset{\tau}{\Longrightarrow} & u \circ p_2 \circ p_{23} \end{array}$$

[5] This data may be assembled into a cube, with X_2 on one vertex, Y on the opposite vertex, three copies of X_1 on vertices adjacent to X_2, three copies of X adjacent to Y, with the various projections along the edges, and the 2-isomorphisms across the sides. To say that $X \to Y$ is a 2-quotient can be thought of as an appropriate "2-cocartesian" property of this cube, which amounts to a descent criterion.

commutes, there is a morphism $v\colon Y \to Z$ and a 2-isomorphism $\rho\colon u \Rightarrow v \circ \pi$ such that the diagram

$$\begin{array}{ccc} u \circ p_1 & \overset{\rho}{\Longrightarrow} & v \circ \pi \circ p_1 \\ {\scriptstyle \tau}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \theta} \\ u \circ p_2 & \underset{\rho}{\Longrightarrow} & v \circ \pi \circ p_2 \end{array}$$

commutes. This must satisfy the following uniqueness property: if $v'\colon Y \to Z$ and $\rho'\colon u \Rightarrow v' \circ \pi$ are another morphism and 2-isomorphism satisfying the same properties, there is a unique 2-isomorphism $\zeta\colon v \Rightarrow v'$ such that the diagram

$$\begin{array}{ccc} u & \overset{\rho}{\Longrightarrow} & v \circ \pi \\ & {\scriptstyle \rho'}\searrow & \Big\Downarrow{\scriptstyle \zeta} \\ & & v' \circ \pi \end{array}$$

commutes.

Definition B.18. The **opposite** 2-category $\mathcal{C}^{\mathrm{op}}$ of a 2-category $\mathcal{C}$ is obtained by reversing the direction of the 1- morphisms, keeping the direction of the 2-morphisms the same. Thus if f and g are morphisms from X to Y in $\mathcal{C}$, and α is a 2-morphism from f to g, then in $\mathcal{C}^{\mathrm{op}}$ there are morphisms f and g from Y to X, with α a 2-morphism from f to g.

3. Adjoints

Adjointness of functors is a familiar notion from category theory. Recall, we say that functors $F\colon \mathcal{X} \to \mathcal{Y}$ and $G\colon \mathcal{Y} \to \mathcal{X}$ are adjoint functors, if for every pair of objects X of $\mathcal{X}$ and Y of $\mathcal{Y}$, we have a bijection

$$\mathrm{Hom}_{\mathcal{X}}(GY, X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{Y}}(Y, FX), \tag{1}$$

natural in X and Y. More specifically, we say that F is right adjoint to G, and G is left adjoint to F. For instance, let $\pi\colon S \to T$ be a continuous map of topological spaces, so we have functors π^{-1} and π_* between the categories of sheaves on S and on T. Then π_* is right adjoint to π^{-1}. In algebraic geometry, when π is a morphism of schemes and our categories are of sheaves of $\mathcal{O}_S$ and $\mathcal{O}_T$ modules, then π_* is right adjoint to π^*; the latter defined by $\pi^*\mathcal{F} = (\pi^{-1}\mathcal{F}) \otimes_{\pi^{-1}\mathcal{O}_T} \mathcal{O}_S$. What is less familiar (and difficult to find in the literature) are results to the effect that adjointness repects base change. The goal of this section is to develop the machinery to arrive at such results in a natural way.

The reader familiar with category theory is probably aware of some equivalent formulations of the notion of adjointness. For instance, the bijection (1) is completely determined by the universal map $Y \to F(GY)$, that is, the image of 1_{GY} under (1) when $X = GY$. There is, similarly, a universal map $G(FX) \to X$. Conversely, a pair of natural transformations $1_{\mathcal{Y}} \Rightarrow F \circ G$ and $G \circ F \Rightarrow 1_{\mathcal{X}}$ satisfying conditions analogous to (2), below, uniquely determines the adjointness relation between F and G. The connections among the various notions of adjointness will be spelled out in detail in section 3.2.

Here we use the language of 2-categories to develop the concept of adjointness and properties relating adjointness with base change. Specializing to the 2-category (Cat), we recover the usual notion of adjoint functors, and specializing further to the adjoint functors π_* and π^* (see Example B.22), we recover the properties concerning adjointness and base change alluded to above.

3.1. Adjunctions. We start with the notion of adjunction, in the form of a pair of functors and universal morphisms, abstracted to a general 2-category.

Definition B.19. Let X and Y be objects in a 2-category $\mathcal{C}$. An **adjunction** from X to Y is a quadruple (f, g, η, ϵ), consisting of two morphisms $f\colon X \to Y$ and $g\colon Y \to X$ and two 2-morphisms $\eta\colon 1_Y \Rightarrow f \circ g$ and $\epsilon\colon g \circ f \Rightarrow 1_X$, such that $(1_f * \epsilon) \circ (\eta * 1_f) = 1_f$ and $(\epsilon * 1_g) \circ (1_g * \eta) = 1_g$; that is, the following diagrams commute:

$$\begin{array}{ccc} 1_Y \circ f & \overset{\eta}{\Longrightarrow} & f \circ g \circ f \\ \Vert & & \Downarrow \epsilon \\ f & = & f \circ 1_X \end{array} \qquad\qquad \begin{array}{ccc} g \circ 1_Y & \overset{\eta}{\Longrightarrow} & g \circ f \circ g \\ \Vert & & \Downarrow \epsilon \\ g & = & 1_X \circ g \end{array} \tag{2}$$

For those who prefer diagrams in 3 dimensions, we can rephrase (2) as the condition that in each diagram below, the "front" faces and "back" faces compose to the same 2-morphism. In the left-hand diagram, the dashed arrow is $f\colon X \to Y$, and in the right-hand diagram it is $g\colon Y \to X$, with the obvious 2-morphisms understood for the "back" faces that border the dashed arrow.

The 2-morphism η is called the **unit** of the adjunction, and ϵ the **counit**.

Exercise B.39. Suppose (f, g, η, ϵ) is an adjunction from X to Y in a 2-category $\mathcal{C}$, and V is any object in $\mathcal{C}$.

(1) If $a\colon V \to X$ and $b\colon V \to Y$ are morphisms in $\mathcal{C}$, there is a canonical bijection

$$\{\text{ 2-morphisms from } g \circ b \text{ to } a\,\} \longleftrightarrow \{\text{ 2-morphisms from } b \text{ to } f \circ a\}.$$

This takes a 2-morphism $\theta\colon g \circ b \Rightarrow a$ to the composite $b = 1_Y \circ b \overset{\eta}{\Rightarrow} f \circ g \circ b \overset{\theta}{\Rightarrow} f \circ a$; the inverse takes a 2-morphism $\pi\colon b \Rightarrow f \circ a$ to the composite $g \circ b \overset{\pi}{\Rightarrow} g \circ f \circ a \overset{\epsilon}{\Rightarrow} 1_X \circ a = a$. Verify that these are inverse bijections.

(2) There is an adjunction $(f^V, g^V, \eta^V, \epsilon^V)$ from $\mathrm{HOM}(V, X)$ to $\mathrm{HOM}(V, Y)$ in the 2-category (Cat). Here f^V, g^V , η^V, and ϵ^V are the functors and natural transformations defined in Exercise B.30, Similarly, there is an adjunction $(g_V, f_V, \eta_V, \epsilon_V)$ from $\mathrm{HOM}(Y, V)$ to $\mathrm{HOM}(X, V)$.

PROPOSITION B.20. *If* (f, g, η, ϵ) *and* $(f, g', \eta', \epsilon')$ *are adjunctions from* X *to* Y*, there is a unique 2-isomorphism* $\theta\colon g \overset{\sim}{\Rightarrow} g'$ *such that* $(1_f * \theta) \circ \eta = \eta'$ *and* $\epsilon' \circ (\theta * 1_f) = \epsilon$*:*

$$
\begin{array}{ccc}
1_Y & \overset{\eta}{\Longrightarrow} & f \circ g \\
& \searrow^{\eta'} & \Downarrow \theta \\
& & f \circ g'
\end{array}
\qquad\qquad
\begin{array}{ccc}
g \circ f & \overset{\epsilon}{\Longrightarrow} & 1_X \\
\theta \Downarrow & \nearrow_{\epsilon'} & \\
g' \circ f & &
\end{array}
$$

PROOF. Define θ to be the composition

$$g = g \circ 1_Y \overset{\eta'}{\Rightarrow} g \circ f \circ g' \overset{\epsilon}{\Rightarrow} 1_X \circ g' = g'.$$

To see that the first diagram commutes, consider the diagram

$$
\begin{array}{ccccccccc}
1_Y & \overset{\eta'}{\Longrightarrow} & f \circ g' & = & 1_Y \circ f \circ g' & = & f \circ g' & & \\
\eta \Downarrow & & & & \Downarrow \eta & & & \diagdown\!\!\diagdown & \\
f \circ g & = & f \circ g \circ 1_Y & \underset{\eta'}{\Longrightarrow} & f \circ g \circ f \circ g' & \underset{\epsilon}{\Longrightarrow} & f \circ 1_X \circ g' & = & f \circ g'
\end{array}
$$

The left rectangle commutes by the exchange property, and the right trapezoid commutes by an adjunction property of η and ϵ. The bottom row is $1_f * \theta$. Similarly, the commutativity of the second diagram is seen from the diagram

$$
\begin{array}{ccccccccc}
g \circ f & = & g \circ 1_Y \circ f & \overset{\eta'}{\Longrightarrow} & g \circ f \circ g' \circ f & \overset{\epsilon}{\Longrightarrow} & 1_X \circ g' \circ f & = & g' \circ f \\
& \diagdown\!\!\diagdown & & & \Downarrow \epsilon' & & & & \Downarrow \epsilon' \\
& & g \circ f & = & g \circ f \circ 1_X & = & g \circ f & \underset{\epsilon}{\Longrightarrow} & 1_X
\end{array}
$$

To see that θ is an isomorphism, define $\theta'\colon g' \Rightarrow g$ to be the composite

$$g' = g' \circ 1_Y \overset{\eta}{\Rightarrow} g' \circ f \circ g \overset{\epsilon'}{\Rightarrow} 1_X \circ g = g.$$

It suffices to show that $\theta' \circ \theta = 1_g$ and $\theta \circ \theta' = 1'_g$. By symmetry, θ' satisfies the identities $(1_f * \theta') \circ \eta' = \eta$ and $\epsilon \circ (\theta' * 1_f) = \epsilon'$. Hence the composite $\theta' \circ \theta$ satisfies the identities $(1_f * (\theta' \circ \theta)) \circ \eta = \eta$ and $\epsilon \circ ((\theta' \circ \theta) * 1_f) = \epsilon$, and similarly for $\theta \circ \theta'$. It therefore suffices to prove the following uniqueness assertion: if $\theta\colon g \Rightarrow g$ satisfies $(1_f * \theta) \circ \eta = \eta$ (and $\epsilon \circ (\theta * 1_f) = \epsilon$), then $\theta = 1_g$. For this, consider the diagram

$$
\begin{array}{ccccccccc}
g & = & g \circ 1_Y & \overset{\eta}{\Longrightarrow} & g \circ f \circ g & \overset{\epsilon}{\Longrightarrow} & 1_X \circ g & = & g \\
\| & & & & \Downarrow 1_g * 1_f * \theta & & \Downarrow \theta & & \Downarrow \theta \\
g & = & g \circ 1_Y & \underset{\eta}{\Longrightarrow} & g \circ f \circ g & \underset{\epsilon}{\Longrightarrow} & 1_X \circ g & = & g
\end{array}
$$

The left rectangle commutes by assumption, the middle square commutes by the exchange property, and the right square commutes by property (c) of 2-categories. Reading around the diagram, one finds $1_g = \theta \circ 1_g$, so $\theta = 1_g$, as required. $\square$

Given a morphism $f\colon X \to Y$ in a 2-category, this proposition justifies the use of the notation $(f, f', \eta^f, \epsilon^f)$ for an adjunction from X to Y, and to call (f', η^f, ϵ^f) (or sometimes just f') a **left adjoint** of f. This notation is particularly useful when we want to compare adjoints for several morphisms. Whenever we have two composable morphisms, each with a left adjoint, the composite can be given the adjoint structure of the following exercise.

EXERCISE B.40. Suppose $(f, f', \eta^f, \epsilon^f)$ is an adjunction from X to Y, and $(g, g', \eta^g, \epsilon^g)$ is an adjunction from Y to Z. Define an adjunction $(g \circ f, f' \circ g', \eta^{gf}, \epsilon^{gf})$ from X to Z, where η^{gf} is the composite

$$1_Z \overset{\eta^g}{\Rightarrow} g \circ g' = g \circ 1_Y \circ g' \overset{\eta^f}{\Rightarrow} g \circ f \circ f' \circ g',$$

and ϵ^{gf} is the composite

$$f' \circ g' \circ g \circ f \overset{\epsilon^g}{\Rightarrow} f' \circ 1_Y \circ f = f' \circ f \overset{\epsilon^f}{\Rightarrow} 1_X.$$

Verify that $(g \circ f, f' \circ g', \eta^{gf}, \epsilon^{gf})$ is an adjunction from X to Z.

3.2. Adjoint functors. When applied to the 2-category (Cat) of categories, the notion of adjunction we have been discussing coincides with the usual notion of adjoint functors. In this context, an adjunction (F, G, η, ϵ) from a category $\mathcal{X}$ to a category $\mathcal{Y}$ consists of functors $F\colon \mathcal{X} \to \mathcal{Y}$, $G\colon \mathcal{Y} \to \mathcal{X}$, and natural transformations $\eta\colon 1_{\mathcal{Y}} \Rightarrow F \circ G$ and $\epsilon\colon G \circ F \Rightarrow 1_{\mathcal{X}}$, such that the composite $F = 1_{\mathcal{Y}} \circ F \overset{\eta}{\Rightarrow} F \circ G \circ F \overset{\epsilon}{\Rightarrow} F \circ 1_{\mathcal{X}} = F$ is the identity on F, and $G = G \circ 1_{\mathcal{Y}} \overset{\eta}{\Rightarrow} G \circ F \circ G \overset{\epsilon}{\Rightarrow} 1_{\mathcal{X}} \circ G = G$ is the identity on G. We say that G is a **left adjoint** of F, and F is a **right adjoint** of G, when this data is specified. If a given F has a left adjoint, it is unique up to a natural isomorphism, by Proposition B.20.

EXERCISE B.41. For every object X of $\mathcal{X}$, $F(\epsilon_X) \circ \eta_{F(X)} = 1_{F(X)}$. For every object Y of $\mathcal{Y}$, $\epsilon_{G(Y)} \circ G(\eta_Y) = 1_{G(Y)}$. For every morphism $a\colon X \to X'$ of $\mathcal{X}$, $a \circ \epsilon_X = \epsilon_{X'} \circ G(F(a))$. For every morphism $b\colon Y \to Y'$ of $\mathcal{Y}$, $F(G(b)) \circ \eta_Y = \eta_{Y'} \circ b$.

The usual definition an adjoint pair of functors prescribes, for every pair of objects $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, a bijection

$$\phi_{Y,X} : \operatorname{Hom}_{\mathcal{X}}(G(Y), X) \to \operatorname{Hom}_{\mathcal{Y}}(Y, F(X)),$$

between the morphisms from $G(Y)$ to X in $\mathcal{X}$ and the morphisms from Y to $F(X)$ in $\mathcal{Y}$, which is natural in X and Y; that is, for any morphisms $a\colon X \to X'$ in $\mathcal{X}$ and $b\colon Y' \to Y$ in $\mathcal{Y}$, the diagrams

$$\begin{array}{ccc} \operatorname{Hom}_{\mathcal{X}}(G(Y), X) & \xrightarrow{\phi_{Y,X}} & \operatorname{Hom}_{\mathcal{Y}}(Y, F(X)) \\ \downarrow{\scriptstyle a_{G(Y)}} & & \downarrow{\scriptstyle F(a)_Y} \\ \operatorname{Hom}_{\mathcal{X}}(G(Y), X') & \xrightarrow[\phi_{Y,X'}]{} & \operatorname{Hom}_{\mathcal{Y}}(Y, F(X')) \end{array} \qquad \begin{array}{ccc} \operatorname{Hom}_{\mathcal{X}}(G(Y), X) & \xrightarrow{\phi_{Y,X}} & \operatorname{Hom}_{\mathcal{Y}}(Y, F(X)) \\ \downarrow{\scriptstyle G(b)^X} & & \downarrow{\scriptstyle b^{F(X)}} \\ \operatorname{Hom}_{\mathcal{X}}(G(Y'), X) & \xrightarrow[\phi_{Y',X}]{} & \operatorname{Hom}_{\mathcal{Y}}(Y', F(X)) \end{array}$$

commute. Equivalently, for $c\colon G(Y) \to X$ in $\mathcal{X}$, and any $a\colon X \to X'$ and $b\colon Y' \to Y$,

$$\phi_{Y',X'}(a \circ c \circ G(b)) \;=\; F(a) \circ \phi_{Y,X}(c) \circ b.$$

These two notions of adjoints coincide, for fixed functors F and G. Given η and ϵ, define $\phi_{Y,X}\colon \operatorname{Hom}(G(Y), X) \to \operatorname{Hom}(Y, F(X))$ by the formula

$$\phi_{Y,X}(c) \;=\; F(c) \circ \eta_Y,$$

i.e., $\phi_{Y,X}(c)$ is the composite $Y \stackrel{\eta_Y}{\to} F(G(Y)) \stackrel{F(c)}{\to} F(X)$. The inverse map from $\operatorname{Hom}(Y, F(X))$ to $\operatorname{Hom}(G(Y), X)$ is defined by

$${\phi_{Y,X}}^{-1}(d) \;=\; \epsilon_X \circ G(d),$$

i.e., ${\phi_{Y,X}}^{-1}(d)$ is the composite $G(Y) \stackrel{G(d)}{\to} G(F(X)) \stackrel{\epsilon_X}{\to} X$. Conversely, given natural bijections $\phi_{Y,X}$ for all X and Y, define η and ϵ by the formulas

$$\eta_Y \;=\; \phi_{Y,G(Y)}(1_{G(Y)}), \quad \epsilon_X \;=\; {\phi_{F(X),X}}^{-1}(1_{F(X)}).$$

Exercise B.42. Verify that the maps $\phi_{Y,X}$ and ${\phi_{Y,X}}^{-1}$ defined from η and ϵ are inverse bijections, natural in X and Y. Verify that the maps η_Y and ϵ_X defined from a collection $\{\phi_{Y,X}\}$ define natural transformations $\eta\colon 1_{\mathcal{Y}} \Rightarrow F \circ G$ and $\epsilon\colon G \circ F \Rightarrow 1_{\mathcal{X}}$, such that (F, G, η, ϵ) defines an adjoint from $\mathcal{X}$ to $\mathcal{Y}$. Verify that these correspondences $\{\phi_{Y,X}\} \leftrightarrow (\eta, \epsilon)$ are inverse bijections.

3.3. Base change.

Definition B.21. Suppose we have a 2-commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle q}\downarrow & \stackrel{\alpha}{\nearrow} & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Z \end{array}$$

in a 2-category, and that each of the morphisms p and q is part of an adjunction $(p, p', \eta^p, \epsilon^p)$ and $(q, q', \eta^q, \epsilon^q)$. Define a **base change** 2-morphism[6]

$$c_\alpha :\; p' \circ f \;\Rightarrow\; g \circ q'$$

to be the composite

$$p' \circ f = p' \circ f \circ 1_X \stackrel{\eta^q}{\Rightarrow} p' \circ f \circ q \circ q' \stackrel{\alpha}{\Rightarrow} p' \circ p \circ g \circ q' \stackrel{\epsilon^p}{\Rightarrow} 1_Y \circ g \circ q' = g \circ q'.$$

If f and g are also part of adjunctions $(f, f', \eta^f, \epsilon^f)$ and $(g, g', \eta^g, \epsilon^g)$, define a 2-morphism

$$\alpha' :\; g' \circ p' \;\Rightarrow\; q' \circ f'$$

to be the composite

$$g' \circ p' = g' \circ p' \circ 1_Z \stackrel{\eta^f}{\Rightarrow} g' \circ p' \circ f \circ f' \stackrel{c_\alpha}{\Rightarrow} g' \circ g \circ q' \circ f' \stackrel{\epsilon^g}{\Rightarrow} 1_W \circ q' \circ f = q' \circ f.$$

[6]In category theory, α and c_α are called *mates* of each other, see [**51**]. Category theorists often write adjunctions in the order (f', f, η, ϵ), with the left adjoint preceding the right adjoint.

Exercise B.43. (1) If p and q have left adjoints, show that the following diagram commutes:

$$\begin{array}{ccccc} f & = & 1_Z \circ f & \overset{\eta^p}{\Longrightarrow} & p \circ p' \circ f \\ \| & & & & \Downarrow c_\alpha \\ f \circ 1_X & \underset{\eta^q}{\Longrightarrow} & f \circ q \circ q' & \underset{\alpha}{\Longrightarrow} & p \circ g \circ q' \end{array}$$

If f and g also have left adjoints, show that the following diagrams commute:

$$\begin{array}{ccccc} p' & = & p' \circ 1_Z & \overset{\eta^f}{\Longrightarrow} & p' \circ f \circ f' \\ \| & & & & \Downarrow c_\alpha \\ 1_Y \circ p' & \underset{\eta^g}{\Longrightarrow} & g \circ g' \circ p' & \underset{\alpha'}{\Longrightarrow} & g \circ q' \circ f' \end{array} \qquad \begin{array}{ccccc} g' \circ p' \circ f & \overset{\alpha'}{\Longrightarrow} & q' \circ f' \circ f & \overset{\epsilon^f}{\Longrightarrow} & q' \circ 1_X \\ c_\alpha \Downarrow & & & & \| \\ g' \circ g \circ q' & \underset{\epsilon^g}{\Longrightarrow} & 1_W \circ q' & = & q' \end{array}$$

(2) Deduce that c_α is equal to the composite

$$p' \circ f = 1_Y \circ p' \circ f \overset{\eta^g}{\Rightarrow} g \circ g' \circ p' \circ f \overset{\alpha'}{\Rightarrow} g \circ q' \circ f' \circ f \overset{\epsilon^f}{\Rightarrow} g \circ q' \circ 1_X = g \circ q'.$$

(3) The correspondence of Exercise B.39, applied to the adjunction $(p, p', \eta^p, \epsilon^p)$ takes $c_\alpha \colon p' \circ f \Rightarrow g \circ q'$ to a 2-morphism from f to $p \circ g \circ q'$. Show that this morphism is the composite $f = f \circ 1_X \overset{\eta^q}{\Rightarrow} f \circ q \circ q' \overset{\alpha}{\Rightarrow} p \circ g \circ q'$. The inverse of the correspondence of Exercise B.39, applied to the adjunction $(g, g', \eta^g, \epsilon^g)$, produces a 2-morphism from $g' \circ p' \circ f$ to q'. Show that this is $g' \circ p' \circ f \overset{\alpha'}{\Rightarrow} q' \circ f' \circ f \overset{\epsilon^f}{\Rightarrow} q'$.

Exercise B.44. (1) Consider a diagram

$$\begin{array}{ccccc} U & \xrightarrow{i} & W & \xrightarrow{g} & Y \\ r\downarrow & \overset{\beta}{\nearrow\!\!\!\nearrow} & \downarrow q & \overset{\alpha}{\nearrow\!\!\!\nearrow} & \downarrow p \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Z \end{array}$$

in a 2-category $\mathcal{C}$. Define $\gamma \colon (f \circ h) \circ r \Rightarrow p \circ (g \circ i)$ to be the composite

$$(f \circ h) \circ r = f \circ h \circ r \overset{\beta}{\Rightarrow} f \circ q \circ i \overset{\alpha}{\Rightarrow} p \circ g \circ i = p \circ (g \circ i).$$

If p, q, and r have left adjoints, show that the diagram

$$\begin{array}{ccccc} p' \circ f \circ h & = & p' \circ (f \circ h) & & \\ c_\alpha \Downarrow & & & \overset{c_\gamma}{\searrow\!\!\!\searrow} & \\ g \circ q' \circ h & \underset{c_\beta}{\Longrightarrow} & g \circ i \circ r' & = & (g \circ i) \circ r' \end{array}$$

commutes.

(2) Dually, given a diagram

$$\begin{array}{ccccc} U & \xrightarrow{i} & W & \xrightarrow{g} & Y \\ {\scriptstyle r}\downarrow & \swArrow_{\beta} & \downarrow{\scriptstyle q} & \swArrow_{\alpha} & \downarrow{\scriptstyle p} \\ V & \xrightarrow[h]{} & X & \xrightarrow[f]{} & Z \end{array}$$

define $\gamma\colon p \circ (g \circ i) \Rightarrow (f \circ h) \circ r$ to be the composite

$$p \circ (g \circ i) = p \circ g \circ i \overset{\alpha}{\Rightarrow} f \circ q \circ i \overset{\beta}{\Rightarrow} f \circ h \circ r = (f \circ h) \circ r.$$

If f, g, h, and i have adjoints, show that the following diagram commutes:

$$\begin{array}{ccccc} h' \circ f' \circ p & = & (f \circ h)' \circ p & & \\ {\scriptstyle c_\alpha}\Downarrow & & & \overset{c_\gamma}{\searrow} & \\ h' \circ q \circ g' & \xRightarrow[c_\beta]{} & r \circ i' \circ g' & = & r \circ (g \circ i)' \end{array}$$

Example B.22. Let

$$\begin{array}{ccc} W & \xrightarrow{g} & Y \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & Z \end{array}$$

be a commutative diagram of schemes. Then there is a natural base change morphism

$$p^* f_* \mathcal{F} \to g_* q^* \mathcal{F}.$$

A detailed treatment of the construction and properties of base change morphisms for sheaves on schemes is given in the Glossary.

4. Pseudofunctors

In this section we consider 2-categories in their natural generality, where one rarely has equality of morphisms; in their place are identities among 2-isomorphisms. Although the definitions and assertions are natural enough, the verifications involve considerable diagram chasing, much of which is left to the interested (and determined) reader.

Definition B.23. If $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, a (covariant) **pseudofunctor** F from $\mathcal{C}$ to $\mathcal{D}$ assigns to each object X of $\mathcal{C}$ an object $F(X)$ of $\mathcal{D}$, to each morphism $f\colon X \to Y$ of $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$, and to each 2-morphism $\alpha\colon f \Rightarrow g$ a 2-morphism $F(\alpha)\colon F(f) \Rightarrow F(g)$. In addition, we must have:

(1) for morphisms $f\colon X \to Y$, $g\colon Y \to Z$ of $\mathcal{C}$, a 2-isomorphism

$$\gamma_{f,g} = \gamma^F_{f,g} : \; F(g \circ f) \overset{\sim}{\Rightarrow} F(g) \circ F(f);$$

(2) for each object X of $\mathcal{C}$, a 2-isomorphism

$$\delta_X = \delta^F_X : \; F(1_X) \overset{\sim}{\Rightarrow} 1_{F(X)}.$$

These must satisfy the following conditions:

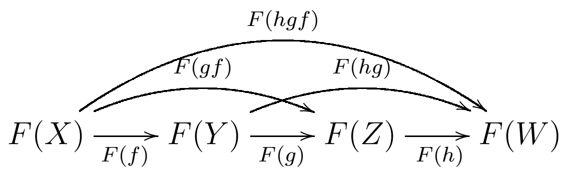
(a) For morphisms $f\colon W \to X$, $g\colon X \to Y$, $h\colon Y \to Z$ in $\mathcal{C}$, we have the equality

$$(1_{F(h)} * \gamma_{f,g}) \circ \gamma_{gf,h} \;=\; (\gamma_{g,h} * 1_{F(f)}) \circ \gamma_{f,hg}$$

of 2-morphisms from $F(h \circ g \circ f)$ to $F(h) \circ F(g) \circ F(f)$:

$$\begin{array}{ccc} F(h \circ g \circ f) & \xRightarrow{\gamma_{f,hg}} & F(h \circ g) \circ F(f) \\ {\scriptstyle \gamma_{gf,h}} \Downarrow & & \Downarrow {\scriptstyle \gamma_{g,h}} \\ F(h) \circ F(g \circ f) & \xRightarrow[\gamma_{f,g}]{} & F(h) \circ F(g) \circ F(f) \end{array}$$

(This can also be described by saying that the two ways to move down in the diagram



agree.)

(b) For a morphism $f\colon X \to Y$ in $\mathcal{C}$, we have the equalities

$$(1_{F(f)} * \delta_X) \circ \gamma_{1_X,f} \;=\; 1_{F(f)} \;=\; (\delta_Y * 1_{F(f)}) \circ \gamma_{f,1_Y}$$

of 2-morphisms from $F(f)$ to $F(f) \circ 1_{F(X)} = F(f) = 1_{F(Y)} \circ F(f)$:

$$\begin{array}{ccc} F(f) & = & F(f) \circ 1_{F(X)} \\ \| & & \Uparrow {\scriptstyle \delta_X} \\ F(f \circ 1_X) & \xRightarrow[\gamma_{1_X,f}]{} & F(f) \circ F(1_X) \end{array} \qquad \begin{array}{ccc} F(f) & = & 1_{F(Y)} \circ F(f) \\ \| & & \Uparrow {\scriptstyle \delta_Y} \\ F(1_Y \circ f) & \xRightarrow[\gamma_{f,1_Y}]{} & F(1_Y) \circ F(f) \end{array}$$

(c) For any morphism f in $\mathcal{C}$, $F(1_f) = 1_{F(f)}$; and, if $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$ in $\mathcal{C}$, we have the equality

$$F(\beta \circ \alpha) = F(\beta) \circ F(\alpha)$$

of 2-morphisms from $F(f)$ to $F(h)$.

(d) If $f, f'\colon X \to Y$, $\alpha\colon f \Rightarrow f'$, $g, g'\colon Y \to Z$, and $\beta\colon g \Rightarrow g'$ in $\mathcal{C}$, then

$$(F(\beta) * F(\alpha)) \circ \gamma_{f,g} \;=\; \gamma_{f',g'} \circ F(\beta * \alpha),$$

an equality of 2-morphisms from $F(g \circ f)$ to $F(g') \circ F(f')$:

$$\begin{array}{ccc} F(g \circ f) & \xRightarrow{F(\beta * \alpha)} & F(g' \circ f') \\ {\scriptstyle \gamma_{f,g}} \Downarrow & & \Downarrow {\scriptstyle \gamma_{f',g'}} \\ F(g) \circ F(f) & \xRightarrow[F(\beta) * F(\alpha)]{} & F(g') \circ F(f') \end{array}$$

We write $F\colon \mathcal{C} \to \mathcal{D}$ to denote that F is a pseudofunctor from $\mathcal{C}$ to $\mathcal{D}$, with associated 2-isomorphisms δ_X^F and $\gamma_{f,g}^F$.

Note by (c) that a pseudofunctor determines (honest) functors

$$\mathrm{HOM}(X, Y) \longrightarrow \mathrm{HOM}(F(X), F(Y))$$

for any objects X and Y in $\mathcal{C}$. Note however that F does *not* induce a functor between the underlying categories of $\mathcal{C}$ and $\mathcal{D}$.

An important special case of this is when $\mathcal{C}$ is an ordinary category, regarded as a 2-category by specifying that its only 2-morphisms are identities. In this case there is no need to specify what F does to 2-morphisms in $\mathcal{C}$, and conditions (c) and (d) can be omitted.

In the text, the situation of a **contravariant pseudofunctor** from an ordinary category $\mathcal{C}$ to a 2-category $\mathcal{D}$ arises. This can be defined to be a pseudofunctor from the opposite category $\mathcal{C}^{\mathrm{op}}$ to $\mathcal{D}$. Explicitly, the changes are: for a morphism $f\colon X \to Y$ one has $F(f)\colon F(Y) \to F(X)$, and for $f\colon X \to Y$, $g\colon Y \to Z$ one has $\gamma_{f,g}\colon F(g \circ f) \overset{\sim}{\Rightarrow} F(f) \circ F(g)$, and the two conditions become:

(a) $(\gamma_{f,g} * 1_{F(h)}) \circ \gamma_{gf,h} \;=\; (1_{F(f)} * \gamma_{g,h}) \circ \gamma_{f,hg}$;
(b) $(\delta_X * 1_{F(f)}) \circ \gamma_{f,1_X} \;=\; 1_{F(f)} \;=\; (1_{F(f)} * \delta_Y) \circ \gamma_{1_Y,f}$.

Exercise B.45. If $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ are pseudofunctors, there is a composite pseudofunctor $G \circ F$. This takes an object X to $G(F(X))$, a morphism f to $G(F(f))$, and a 2-morphism ρ to $G(F(\rho))$; and one sets $\gamma^{G\circ F}_{f,g} = \gamma^{G}_{F(f),F(g)} \circ G(\gamma^{F}_{f,g})$ and $\delta^{GF}_X = \delta^{G}_{F(X)} \circ G(\delta^{F}_X)$. Verify that this defines a pseudofunctor. Show that 2-categories, with pseudofunctors as morphisms, form a category.

Exercise B.46. Construct a pseudofunctor B from the 2-category (Grp) of groups to the 2-category (Cat) that takes a group G to the category BG of G-torsors (where a group G is regarded as a topological group with the discrete topology).

Definition B.24. If F and G are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, a **pseudonatural transformation** α from F to G consists of

(1) For each object X in $\mathcal{C}$, a morphism $\alpha_X\colon F(X) \to G(X)$ in $\mathcal{D}$.
(2) For each morphism $f\colon X \to Y$ in $\mathcal{C}$, a 2-isomorphism

$$\tau_f = \tau^{\alpha}_f :\; G(f) \circ \alpha_X \;\overset{\sim}{\Rightarrow}\; \alpha_Y \circ F(f)$$

in $\mathcal{D}$. This is displayed in the diagram

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \alpha_X}\downarrow & {\scriptstyle \tau_f}\Nearrow & \downarrow{\scriptstyle \alpha_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

These must satisfy:

(a) For morphisms $f\colon X \to Y$, $g\colon Y \to Z$ of $\mathcal{C}$, we have an equality

$$(1_{\alpha_Z} * \gamma^{F}_{f,g}) \circ \tau_{gf} \;=\; (\tau_g * 1_{F(f)}) \circ (1_{G(g)} * \tau_f) \circ (\gamma^{G}_{f,g} * 1_{\alpha_X})$$

of 2-morphisms from $G(gf)\circ\alpha_X$ to $\alpha_Z\circ F(g)\circ F(f)$:

$$\begin{array}{ccccc}
G(g\circ f)\circ\alpha_X & \overset{\gamma^G_{f,g}}{\Longrightarrow} & G(g)\circ G(f)\circ\alpha_X & \overset{\tau_f}{\Longrightarrow} & G(g)\circ\alpha_Y\circ F(f)\\
{\scriptstyle\tau_{gf}}\Downarrow & & & & \Downarrow{\scriptstyle\tau_g}\\
\alpha_Z\circ F(g\circ f) & & \underset{\gamma^F_{f,g}}{\Longrightarrow} & & \alpha_Z\circ F(g)\circ F(f)
\end{array}$$

(b) For any object X of $\mathcal{C}$, we have the equality

$$(1_{\alpha_X}*\delta^F_X)\circ\tau_{1_X} \;=\; \delta^G_X*1_{\alpha_X}$$

of 2-morphisms from $G(1_X)\circ\alpha_X$ to $\alpha_X\circ 1_{F(X)}=\alpha_X=1_{G(X)}\circ\alpha_X$:

$$\begin{array}{ccc}
G(1_X)\circ\alpha_X & \overset{\delta^G_X}{\Longrightarrow} & 1_{G(X)}\circ\alpha_X\\
{\scriptstyle\tau_{1_X}}\Downarrow & & \|\\
\alpha_X\circ F(1_X) & \underset{\delta^F_X}{\Longrightarrow} & \alpha_X\circ 1_{F(X)}
\end{array}$$

(c) For $f,g\colon X\to Y$, and a 2-morphism $\rho\colon f\Rightarrow g$ in $\mathcal{C}$, we have the equality

$$\tau_g\circ(G(\rho)*1_{\alpha_X}) \;=\; (1_{\alpha_Y}*F(\rho))\circ\tau_f$$

of 2-morphisms from $G(f)\circ\alpha_X$ to $\alpha_Y\circ F(g)$:

$$\begin{array}{ccc}
G(f)\circ\alpha_X & \overset{\tau_f}{\Longrightarrow} & \alpha_Y\circ F(f)\\
{\scriptstyle G(\rho)}\Downarrow & & \Downarrow{\scriptstyle F(\rho)}\\
G(g)\circ\alpha_X & \underset{\tau_g}{\Longrightarrow} & \alpha_Y\circ F(g)
\end{array}$$

Exercise B.47. If F,G,H are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, one can compose pseudonatural transformations α from F to G and β from G to H to get a pseudonatural transformation $\beta\circ\alpha$ from F to H. This is defined by setting $(\beta\circ\alpha)_X=\beta_X\circ\alpha_X$ and $\tau^{\beta\circ\alpha}_f=(1_{\beta_Y}*\tau^\alpha_f)\circ(\tau^\beta_f*1_{\alpha_X})$. Show that this composition is associative, and has identities, so that the pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ and the pseudonatural transformations between them form the objects and morphisms of a category.

Definition B.25. Suppose $\mathcal{C}$ and $\mathcal{D}$ are 2-categories, F and G are pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$, and α and β are pseudonatural transformations from F to G. A **modification** Θ from α to β assigns to each object X of $\mathcal{C}$ a 2-morphism $\Theta_X\colon\alpha_X\Rightarrow\beta_X$:

$$F(X)\;\overset{\alpha_X}{\underset{\beta_X}{\rightrightarrows}}\;G(X)\qquad(\Downarrow\Theta_X).$$

This must satisfy the property that for any morphisms $f,g\colon X\to Y$ and 2-morphism $\rho\colon f\Rightarrow g$ in $\mathcal{C}$, we have the equality

$$(\Theta_Y*F(\rho))\circ\tau^\alpha_f \;=\; \tau^\beta_g\circ(G(\rho)*\Theta_X)$$

of 2-morphisms from $G(f) \circ \alpha_X$ to $\beta_Y \circ F(g)$:

$$\begin{array}{ccc} G(f) \circ \alpha_X & \overset{G(\rho)*\Theta_X}{\Longrightarrow} & G(g) \circ \beta_X \\ {\scriptstyle \tau_f^\alpha}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \tau_g^\beta} \\ \alpha_Y \circ F(f) & \underset{\Theta_Y * F(\rho)}{\Longrightarrow} & \beta_Y \circ F(g) \end{array}$$

We write $\Theta\colon \alpha \Rrightarrow \beta$ to indicate that Θ is a modification from α to β. We call a modification an **isomodification** if each Θ_X is a 2-isomorphism. In this case we write $\Theta\colon \alpha \overset{\sim}{\Rrightarrow} \beta$.

Note that each of the conditions on pseudofunctors, pseudonatural transformations, and modifications is stated as an equality of 2-morphisms.

Exercise B.48. Show that the property of a modification in Definition B.25 follows from the property

$$(\Theta_Y * 1_{F(f)}) \circ \tau_f^\alpha \;=\; \tau_f^\beta \circ (1_{G(f)} * \Theta_X)$$

for any morphism $f\colon X \to Y$ in $\mathcal{C}$.

Exercise B.49. If $\Theta\colon \alpha \Rrightarrow \beta$ and $\Xi\colon \beta \Rrightarrow \gamma$, with α, β, and γ pseudonatural transformations from F to G, there is a modification $\Xi\circ\Theta\colon \alpha \Rrightarrow \gamma$, defined by $(\Xi\circ\Theta)_X = \Xi_X \circ \Theta_X$. If $\Theta\colon \alpha \Rrightarrow \alpha'$, with $\alpha, \alpha'\colon F \Rightarrow G$, and $\Xi\colon \beta \Rrightarrow \beta'$, with $\beta, \beta'\colon G \Rightarrow H$, there is a modification $\Xi * \Theta\colon \beta \circ \alpha \Rrightarrow \beta' \circ \alpha'$, defined by $(\Xi * \Theta)_X = \Xi_X * \Theta_X$. For fixed 2-categories $\mathcal{C}$ and $\mathcal{D}$, these operations, together with those of Exercise B.47, make the pseudofunctors from $\mathcal{C}$ to $\mathcal{D}$ into the objects of a 2-category $\mathrm{PSFUN}(\mathcal{C}, \mathcal{D})$, with arrows given by pseudonatural transformations, and 2-cells given by modifications.

When α and β are 2-natural transformations between 2-functors F and G, the condition on a modification simplifies to the equation $\Theta_Y * F(\rho) = G(\rho) * \Theta_X$.

Exercise B.50. $^{(*)}$ Given modifications $\Theta\colon \alpha \Rrightarrow \alpha'$ and $\Xi\colon \beta \Rrightarrow \beta'$ between 2-natural transformations $\alpha, \alpha'\colon F \Rightarrow F'$, $\beta, \beta'\colon G \Rightarrow G'$, with 2-functors $F, F'\colon \mathcal{C} \to \mathcal{D}$, $G, G'\colon \mathcal{D} \to \mathcal{E}$, define a modification $\Xi \diamond \Theta\colon \beta * \alpha \Rrightarrow \beta' * \alpha'$ by the formula

$$(\Xi \diamond \Theta)_X = G'(\Theta_X) * \Xi_{F(X)} = \Xi_{F'(X)} * G(\Theta_X).$$

Show that 2-categories, 2-functors, 2-natural transformations, and modifications form the objects, arrows, 2-cells, and 3-cells of a **3-category**: in addition to the underlying 2-category structure formed by the objects, arrows, and 2-cells, the three operations $\circ$, $*$, and $\diamond$ on 3-cells satisfy the following associativity, exchange, and unity identities:[7]

(a) $\Gamma \circ (\Xi \circ \Theta) = (\Gamma \circ \Xi) \circ \Theta$, $\Gamma * (\Xi * \Theta) = (\Gamma * \Xi) * \Theta$, and $\Gamma \diamond (\Xi \diamond \Theta) = (\Gamma \diamond \Xi) \diamond \Theta$;

(b) $(\Xi' \circ \Xi) * (\Theta' \circ \Theta) = (\Xi' * \Theta') \circ (\Xi * \Theta)$, $(\Xi' * \Xi) \diamond (\Theta' * \Theta) = (\Xi' \diamond \Theta') * (\Xi \diamond \Theta)$, $(\Xi' \circ \Xi) \diamond (\Theta' \circ \Theta) = (\Xi' \diamond \Theta') \circ (\Xi \diamond \Theta)$;

[7] In each identity it is assumed that one, and hence the other, side of the equation is defined.

(c) each 2-cell α has an identity 3-cell $1_\alpha\colon \alpha \Rrightarrow \alpha$, and the following identities are satisfied: $1_\beta * 1_\alpha = 1_{\beta\circ\alpha}$ when $\beta \circ \alpha$ is defined; $1_\beta \diamond 1_\alpha = 1_{\beta * \alpha}$ when $\beta * \alpha$ is defined; in addition, if $\Theta\colon \alpha \Rrightarrow \beta$, $\alpha, \beta\colon f \Rightarrow g$, $f, g\colon X \to Y$, then

$$\Theta \circ 1_\alpha = \Theta = 1_\beta \circ \Theta, \Theta * 1_{1_f} = \Theta = 1_{1_g} * \Theta, \Theta \diamond 1_{1_{1_X}} = \Theta = 1_{1_{1_Y}} \diamond \Theta.$$

A formal definition of 3-categories (equivalent to that in the preceding exercise) can be found in [**12**], §7.3, but note that 2-categories, pseudofunctors, and pseudonatural transformations, and modifications do *not* form a 3-category, only something weaker called a (Gray) tricategory [**32**]. In fact, in contrast with Exercise B.36, 2-categories, pseudofunctors, and pseudonatural transformations do not form a 2-category. In any case, we have will have no need for the formalism of 3-categories.

Note that a 2-functor F is just a pseudofunctor for which the associated 2-isomorphisms $\gamma^F_{f,g}$ and δ^F_X are identities. A 2-natural transformation $\alpha\colon F \Rightarrow G$ between 2-functors is a pseudonatural transformation such that τ^α_f is an identity for all morphisms f.

Exercise B.51. For fixed 2-categories $\mathcal{C}$ and $\mathcal{D}$, the 2-functors, 2-natural transformations, and modifications determine a sub-2-category 2-FUN$(\mathcal{C}, \mathcal{D})$ of the 2-category PSFUN$(\mathcal{C}, \mathcal{D})$.

Exercise B.52. For any 2-category $\mathcal{C}$, construct a 2-functor

$$\mathcal{C} \to \text{2-FUN}(\mathcal{C}^{\text{op}}, (\text{Cat})).$$

The following theorem gives the notion of "isomorphism" between 2-categories that one is likely to meet in practice.

Theorem B.26. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a pseudofunctor between 2-categories. The following are equivalent:*

(1) (i) *Every object of $\mathcal{D}$ is 2-isomorphic to an object of the form $F(P)$ for some object P in $\mathcal{C}$;* (ii) *For any objects P and Q in $\mathcal{C}$, every morphism from $F(P)$ to $F(Q)$ is 2-isomorphic to a morphism of the form $F(a)$, for some morphism $a\colon P \to Q$ in $\mathcal{C}$;* (iii) *For morphisms a and b from P to Q in $\mathcal{C}$, any 2-morphism from $F(a)$ to $F(b)$ has the form $F(\rho)$ for a unique 2-morphism $\rho\colon a \Rightarrow b$ in $\mathcal{C}$.*

(2) *There is a pseudofunctor $G\colon \mathcal{D} \to \mathcal{C}$, together with four pseudonatural transformations:*

$$\alpha\colon G \circ F \Rightarrow 1_{\mathcal{C}}, \quad \alpha'\colon 1_{\mathcal{C}} \Rightarrow G \circ F, \quad \beta\colon F \circ G \Rightarrow 1_{\mathcal{D}}, \quad \beta'\colon 1_{\mathcal{D}} \Rightarrow F \circ G,$$

and four isomodifications:

$$\Theta\colon 1_{FG} \overset{\sim}{\Rrightarrow} \beta' \circ \beta, \quad \Theta'\colon 1_{1_{\mathcal{D}}} \overset{\sim}{\Rrightarrow} \beta \circ \beta', \quad \Xi\colon 1_{GF} \overset{\sim}{\Rrightarrow} \alpha' \circ \alpha, \quad \Xi'\colon 1_{1_{\mathcal{C}}} \overset{\sim}{\Rrightarrow} \alpha \circ \alpha'.$$

(3) *As in (2), but with the additional identities:*

$$1_\beta * \Theta = \Theta' * 1_\beta, \quad 1_{\beta'} * \Theta' = \Theta * 1_{\beta'}, \quad 1_\alpha * \Xi = \Xi' * 1_\alpha, \quad 1_{\alpha'} * \Xi' = \Xi * 1_{\alpha'}.$$

(These are modifications from β to $\beta \circ \beta' \circ \beta$, from β' to $\beta' \circ \beta \circ \beta'$, from α to $\alpha \circ \alpha' \circ \alpha$, and from α' to $\alpha' \circ \alpha \circ \alpha'$, respectively.)

Note that conditions (ii) and (iii) of (1) say that every induced functor $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ is a an equivalence of categories.

Proof. This is at least a folk theorem in 2-category theory, cf. [**62**, §2.2], but since it is not easy to find a reference, we will sketch a proof. That (2) implies (1) is quite straightforward. First we show that (2) implies (i) of (1). Since Θ_Y is a 2-isomorphism $1_{FG(Y)} \Rightarrow \beta'_Y \circ \beta_Y$ and Θ'_Y is a 2-isomorphism $1_Y \Rightarrow \beta_Y \circ \beta'_Y$, it follows that β_Y and β'_Y are 2-equivalences, for any object Y of $\mathcal{D}$. In particular, Y is 2-isomorphic to $F(G(Y))$. Note, similarly, that α_X and α'_X are 2-equivalences, for any object X of $\mathcal{C}$. Now we show the remainder of (2) $\Rightarrow$ (1), namely, that every functor $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ determined by F is an equivalence of categories. Following this by the functor $\mathrm{HOM}(F(P),F(Q)) \to \mathrm{HOM}(GF(P),GF(Q))$ induced by G, and then by the functors

$$\mathrm{HOM}(GF(P),GF(Q)) \xrightarrow{(\alpha'_P)_{GF(Q)}} \mathrm{HOM}(P,GF(Q)) \xrightarrow{(\alpha_Q)^P} \mathrm{HOM}(P,Q),$$

each of which is an equivalence of categories by Proposition B.10 and Exercise B.31, one obtains a functor from $\mathrm{HOM}(P,Q)$ to itself. A natural isomorphism of this functor with the identity functor is given by sending $f\colon P \to Q$ to

$$\alpha_Q \circ GF(f) \circ \alpha'_P \overset{\tau_f^{\alpha'}}{\Rightarrow} \alpha_Q \circ \alpha'_Q \circ f \overset{\Xi'^{-1}_Q}{\Rightarrow} 1_Q \circ f = f;$$

the naturality is proved by a use of property (c) for the pseudonatural transformation α'. It follows that $\mathrm{HOM}(P,Q) \to \mathrm{HOM}(F(P),F(Q))$ is full and faithful, and, by the same applied to G, that $\mathrm{HOM}(F(P),F(Q)) \to \mathrm{HOM}(GF(P),GF(Q))$ is also full and faithful; it then follows from Exercise B.9 that each of them must also be an equivalence of categories.

We will show how to use (1) to construct all the data needed for (3). Then 25 identities among 2-isomorphisms must be verified to prove (3): 7 to prove that G is a pseudofunctor, 3 each to prove that α, α', β, and β' are pseudonatural transformations, 4 to verify that Θ, Θ', Ξ, and Ξ' are modifications, and 4 for the last conditions stated in (3). A few of these identities will be immediate from the construction, but most require – at least without a more sophisticated categorical language – tracing around rather large diagrams in various HOM categories. The key that makes it all work is a careful use of Proposition B.10.

For each object X of $\mathcal{D}$, use (i) and Proposition B.10 to choose an object $G(X)$ in $\mathcal{C}$ together with morphisms $\beta_X\colon F(G(X)) \to X$ and $\beta'_X\colon X \to F(G(X))$, and with 2-isomorphisms $\Theta_X\colon 1_{FG(X)} \overset{\sim}{\Rightarrow} \beta'_X \circ \beta_X$ and $\Theta'_X\colon 1_X \overset{\sim}{\Rightarrow} \beta_X \circ \beta'_X$, such that the two conditions

$$1_{\beta_X} * \Theta_X = \Theta'_X * 1_{\beta_X} \quad \text{and} \quad 1_{\beta'_X} * \Theta'_X = \Theta_X * 1_{\beta'_X}$$

are satisfied. For each morphism $f\colon X \to Y$ in $\mathcal{D}$, use (ii) to choose a morphism $G(f)\colon G(X) \to G(Y)$ in $\mathcal{C}$, together with a 2-isomorphism $\lambda_f\colon F(G(f)) \overset{\sim}{\Rightarrow} \beta'_Y \circ f \circ \beta_X$. Now for morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{D}$, use (iii) to define $\gamma^G_{f,g}\colon G(g \circ f) \overset{\sim}{\Rightarrow} G(g) \circ G(f)$, by requiring $F(\gamma^G_{f,g})$ to make the following diagram commute:

$$\begin{array}{ccccc}
FG(g\circ f) & \xRightarrow{F(\gamma^G_{f,g})} & F(G(g)\circ G(f)) & \xRightarrow{\gamma^F_{G(f),G(g)}} & FG(g)\circ FG(f) \\
{\scriptstyle \lambda_{gf}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \lambda_g * \lambda_f} \\
\beta'_Z\circ g\circ f\circ\beta_X & = & \beta'_Z\circ g\circ 1_Y\circ f\circ\beta_X & \xRightarrow[\Theta'_Y]{} & \beta'_Z\circ g\circ\beta_Y\circ\beta'_Y\circ f\circ\beta_X
\end{array}$$

For each object X in $\mathcal{D}$, define $\delta^G_X\colon G(1_X) \overset{\sim}{\Rightarrow} 1_{G(X)}$ by requiring $F(\delta^G_X)$ to make the following diagram commute:

$$\begin{array}{ccc}
FG(1_X) & \xRightarrow{F(\delta^G_X)} & F(1_{G(X)}) \\
{\scriptstyle \lambda_{1_X}}\Big\Downarrow & & \Big\Downarrow{\scriptstyle \delta^F_{G(X)}} \\
\beta'_X\circ\beta_X & \xLeftarrow[\Theta_X]{} & 1_{FG(X)}
\end{array}$$

For $f\colon X\to Y$ in $\mathcal{D}$, define $\tau^\beta_f\colon f\circ\beta_X \overset{\sim}{\Rightarrow} \beta_Y\circ FG(f)$ and $\tau^{\beta'}_f\colon FG(f)\circ\beta'_X \overset{\sim}{\Rightarrow} \beta'_Y\circ f$ to make the following diagrams commute:

$$\begin{array}{ccc}
f\circ\beta_X & \xRightarrow{\tau^\beta_f} & \beta_Y\circ FG(f) \\
\Big\Vert & & \Big\Downarrow{\scriptstyle \lambda_f} \\
1_Y\circ f\circ\beta_X & \xRightarrow[\Theta'_Y]{} & \beta_Y\circ\beta'_Y\circ f\circ\beta_X
\end{array}
\qquad
\begin{array}{ccc}
FG(f)\circ\beta'_X & \xRightarrow{\tau^{\beta'}_f} & \beta'_Y\circ f \\
{\scriptstyle \lambda_f}\Big\Downarrow & & \Big\Vert \\
\beta'_Y\circ f\circ\beta_X\circ\beta'_X & \xLeftarrow[\Theta'_X]{} & \beta'_Y\circ f\circ 1_X
\end{array}$$

For each object P in $\mathcal{C}$, use (ii) to choose a morphism $\alpha_P\colon GF(P)\to P$, together with a 2-isomorphism $\mu_P\colon F(\alpha_P) \overset{\sim}{\Rightarrow} \beta_{F(P)}$. For each morphism $a\colon P\to Q$ in $\mathcal{C}$, $\tau^\alpha_a\colon a\circ\alpha_P \overset{\sim}{\Rightarrow} \alpha_Q\circ GF(a)$ is determined so $F(\tau^\alpha_a)$ makes

$$\begin{array}{ccccc}
F(a\circ\alpha_P) & \xRightarrow{F(\tau^\alpha_a)} & F(\alpha_Q\circ GF(a)) & \xRightarrow{\gamma^F_{GF(a),\alpha_Q}} & F(\alpha_Q)\circ FGF(a) \\
{\scriptstyle \gamma^F_{\alpha_P,a}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \mu_Q} \\
F(a)\circ F(\alpha_P) & \xRightarrow[\mu_P]{} & F(a)\circ\beta_{F(P)} & \xRightarrow[\tau^\beta_{F(a)}]{} & \beta_{F(Q)}\circ FGF(a)
\end{array}$$

commute. Similarly choose $\alpha'_P\colon P\to GF(P)$ with $\mu'_P\colon F(\alpha'_P) \overset{\sim}{\Rightarrow} \beta'_{F(P)}$, and determine $\tau^{\alpha'}_a\colon GF(a)\circ\alpha'_P \overset{\sim}{\Rightarrow} \alpha'_Q\circ a$ so $F(\tau^{\alpha'}_a)$ makes

$$\begin{array}{ccccc}
F(GF(a)\circ\alpha'_P) & \xRightarrow{F(\tau^{\alpha'}_a)} & F(\alpha'_Q\circ a) & \xRightarrow{\gamma^F_{a,\alpha'_Q}} & F(\alpha'_Q)\circ F(a) \\
{\scriptstyle \gamma^F_{\alpha'_P,GF(a)}}\Big\Downarrow & & & & \Big\Downarrow{\scriptstyle \mu'_Q} \\
FGF(a)\circ F(\alpha'_P) & \xRightarrow[\mu'_P]{} & FGF(a)\circ\beta'_{F(P)} & \xRightarrow[\tau^{\beta'}_{F(a)}]{} & \beta'_{F(Q)}\circ F(a)
\end{array}$$

commute. For $f, g\colon X \to Y$, and $\rho\colon f \Rightarrow g$ in $\mathcal{D}$, define $G(\rho)\colon G(f) \Rightarrow G(g)$ by requiring that $F(G(\rho))$ makes the diagram

$$\begin{array}{ccc} FG(f) & \xrightarrow{F(G(\rho))} & FG(g) \\ \lambda_f \Big\Downarrow & & \Big\Downarrow \lambda_f \\ \beta'_Y \circ f \circ \beta_X & \underset{\rho}{\Longrightarrow} & \beta'_Y \circ g \circ \beta_X \end{array}$$

commute. Finally, define, for each object P in $\mathcal{C}$, 2-isomorphisms $\Xi_P\colon 1_{GF(P)} \overset{\sim}{\Rightarrow} \alpha'_P \circ \alpha_P$ and $\Xi'_P\colon 1_P \overset{\sim}{\Rightarrow} \alpha_P \circ \alpha'_P$, determined by the commutativity of the diagrams

$$\begin{array}{ccccc} F(1_{GF(P)}) & \xrightarrow{F(\Xi_P)} & F(\alpha'_P \circ \alpha_P) & \xrightarrow{\gamma^F_{\alpha_P,\alpha'_P}} & F(\alpha'_P) \circ F(\alpha_P) \\ \delta^F_{GF(P)} \Big\Downarrow & & & & \Big\Downarrow \mu'_P * \mu_P \\ 1_{FGF(P)} & & \underset{\Theta_{F(P)}}{\Longrightarrow} & & \beta'_{F(P)} \circ \beta_{F(P)} \end{array}$$

and

$$\begin{array}{ccccc} F(1_P) & \xrightarrow{F(\Xi'_P)} & F(\alpha_P \circ \alpha'_P) & \xrightarrow{\gamma^F_{\alpha'_P,\alpha_P}} & F(\alpha_P) \circ F(\alpha'_P) \\ \delta^F_P \Big\Downarrow & & & & \Big\Downarrow \mu_P * \mu'_P \\ 1_{F(P)} & & \underset{\Theta'_{F(P)}}{\Longrightarrow} & & \beta_{F(P)} \circ \beta'_{F(P)} \end{array}$$

This completes the construction of the data. To prove each of the required identities, one writes it as a diagram, in which the maps are 2-isomorphisms between two morphisms, usually in $\mathcal{C}$, that should commute. By the faithfulness of F on the HOM categories, it suffices to prove this after applying F. One then uses the diagrams just constructed to see what this means, obtaining a large diagram that should commute. Finally, one finds a way to subdivide this large diagram into smaller diagrams that commute by properties of F and properties of 2-categories, especially the exchange property. □

Exercise B.53. Complete the proof of this proposition.

Definition B.27. A pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ is a **pseudoequivalence** if it satisfies the equivalent conditions of the proposition.

Exercise B.54. Show that the composite of two pseudoequivalences is a pseudoequivalence. Being pseudoequivalent is therefore an equivalence relation on 2-categories.

There are several generalizations of these notions and results. The conditions on pseudofunctors F and pseudonatural transformations α can be weakened, by allowing the associated $\gamma^F_{f,g}$, δ^F_X, and τ^α_f (or sometimes their inverses) to be only 2-morphisms, not 2-isomorphisms. Such are called *lax functors* and *lax natural transformations*. There is also a notion of a *bicategory*, in which the associativity equality $\gamma \circ (\beta \circ \alpha) = (\gamma \circ \beta) \circ \alpha$ of morphisms is replaced by a 2-isomorphism, and similarly for the equalities $f = f \circ 1_X =$

$1_Y \circ f$ for a morphism $f\colon X \to Y$. (In a bicategory, each $\mathrm{HOM}(X,Y)$ is a category, but removing the 2-cells from a bicategory does not leave a category.) Theorem B.26 generalizes to the setting of bicategories, cf. [**62**]. Bicategories, 3-categories, and lax notions are discussed in [**12**, §7].

5. Two remarks on set theory

The reader will have noticed that our discussion of categories is framed entirely in the language of "naive set theory" — indeed, sets were not mentioned at all, except in examples. In naive set theory, a set is prescribed by knowing the elements in the set. In this sense, the objects and morphisms of a category (and the 2-morphisms of a 2-category) form sets, since these objects, morphisms (and 2-morphisms) are assumed to be precisely defined. Thus a category is a permissible object in a category, since we know what a category is, and so we have the category (Cat) of categories, which is a 2-category (cf. B.8). And, of course, this leads to Bertrand Russell's famous paradox about the set of all sets.

This problem is not particular to stacks, and those who have made peace with this problem in other areas need not fret about it when studying stacks.[8] The fact that category theory plays a prominent role in the study of stacks, however, brings these problems closer to the surface. And, just as going from objects to categories causes the problem about the set of all sets, going from ordinary categories to 2-categories increases the difficulty. Our aim in this section is to discuss these problems briefly, and point out how to get around them. In this setting, one is not working with sets as collections of well-defined objects, but within an axiomatic set theory, usually taken today to be Zermelo-Fraenkel theory. We also discuss our use of the axiom of choice.

5.1. Categories. We have the need to consider three basic types of set-theoretic structures: sets, categories and 2-categories. If we would like to consider the category of all sets, for example, it is clear that categories cannot be sets. Similarly, if we need to consider the 2-category of all categories, then 2-categories cannot be categories. In spite of Russell's paradox, we would like to do all usual set-theoretic operations (products, disjoint unions, power sets etc.) not only with sets, but also with categories (less so with 2-categories). For this it is necessary to introduce some set-theoretic hierarchy.

We may require all sets to be sets in the usual sense, i.e., the sets given by the axioms of Zermelo-Fraenkel. Then we postulate the existence of *classes* and require categories to be classes. More precisely, the collection of all morphisms in a category is required to be a class — after all, a category can be thought of as its collection of morphisms with a partially defined binary operation, the objects are then defined in terms of the identity morphisms. We require classes to satisfy the same list of axioms as sets. Besides the existence of sets and classes, we moreover postulate the existence of *2-classes* and require 2-categories to be 2-classes; note that 2-categories may be identified with their collection of 2-arrows.

[8] We quote the footnote on the first page of the revised version [**EGA** I′]: "Nous considérons les catégories d'un point de vue 'naif', comme s'il s'agissait d'ensembles et renvoyons à SGA, 4, I pour les questions de logique liées à la théorie des catégories, et la justification du langage que nous utilisons."

Alternatively, (more precisely?) the required set-theoretic hierarchy may be introduced via the theory of *universes*, cf. [**65**], §I.6, and [SGA4] §I. One postulates the existence of three universes, $\mathfrak{U}_0$, $\mathfrak{U}_1$ and $\mathfrak{U}_2$ and requires that $\mathfrak{U}_0 \in \mathfrak{U}_1 \in \mathfrak{U}_2$. Recall that all universes are actually sets (and thus consist of sets). We then use all sets in $\mathfrak{U}_0$ as sets, all sets in $\mathfrak{U}_1$ as classes and all sets in $\mathfrak{U}_2$ as 2-classes. In other words, we require all sets (as the term is used in the book) to be in $\mathfrak{U}_0$, all categories to be in $\mathfrak{U}_1$ and all 2-categories to be in $\mathfrak{U}_2$. If a category happens to be in $\mathfrak{U}_0$, we call it **small**.

For many applications this notion of small is too restrictive. We find much more useful the notion of *essential* smallness. A category is called **essentially small** if its collection of isomorphism classes (i.e. its collection of objects up to isomorphism) is in $\mathfrak{U}_0$, i.e. is a set. When fibered categories are introduced, one can require the fibers to be essentially small. These and the other groupoids that appear can be taken to be essentially small.

We will not mention these set-theoretic issues in the main body of the book. The reader who thinks in this language is invited to check that none of our constructions with categories leave the universe $\mathfrak{U}_1$, and that every time we construct a set from a category we obtain a set in $\mathfrak{U}_0$.

5.2. The axiom of choice. The only essential way in which we use the axiom of choice is the following: we want a functor which is faithful, full, and essentially surjective to have an inverse (a functor in the other direction such that both compositions are naturally isomorphic to the identities). This is important because the natural notion of isomorphism between stacks is expressed by such an equivalence. Note that this actually uses the axiom of choice for *classes* of sets. This use of the axiom of choice can be avoided if one understands that whenever we call two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ *equivalent* , this means that there exists a chain of categories and equivalences

$$\mathcal{C}_1 \longleftarrow \mathcal{D}_1 \longrightarrow \mathcal{D}_2 \longleftarrow \ldots \longleftarrow \mathcal{D}_n \longrightarrow \mathcal{C}_2 \quad .$$

In effect, this amounts to localizing the 2-category of categories at the equivalences (For a discussion of similar localizations, see [**28**]). We prefer not to do this, but rather assume that all equivalences have inverses. In view of these remarks, this use of the axiom of choice may be viewed as merely a device of notational convenience.

We also remark that even though many general statements in the book formally require the axiom of choice for their validity, in any specific application one usually has more or less canonical choices at one's disposal. For example, the fibered product of schemes may be constructed in a canonical way, cf. [**EGA** I]; we do not need the axiom of choice to pick an object satisfying the universal mapping property of fibered product. Similarly, the construction of quotient sheaves can be carried out in a canonical way by sheafifying (using an explicit sheafification construction) a given quotient presheaf.

In short, the reader can choose whatever set-theoretic foundations he or she is comfortable with: we do not discuss them in the text. As with other areas of algebra or geometry, the notions and theorems about stacks change very little with changes in logical foundations.

Answers to Exercises

B.1. If $f \circ a = f$ for all f, and $b \circ g = g$ for all g, then $a = b \circ a = b$.

B.2. (1) If $g \circ f = 1_X$ and $f \circ h = 1_Y$, then $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$. (2) follows similarly from associativity.

B.7. For $\Rightarrow$, given G, θ, and η, a morphism $f\colon F(P) \to F(Q)$ has uniquely the form $f = F(a)$, where $a\colon P \to Q$ is given by $a = \theta_Q \circ G(f) \circ \theta_P^{-1}$. For any object X in $\mathcal{D}$, the isomorphism $\eta_X\colon FG(X) \to X$ shows that F is essentially surjective. Details can be found in [**65**, §IV.4].

B.8. This is exactly what the proof of the proposition produces.

B.10. Take $X \times_Z Y = \{(x, y) \in X \times Y \mid s(x) = t(y)\}$, with the induced topology in the topological case.

B.13. A natural transformation θ from h_X to H assigns to every object S of $\mathcal{C}$ a mapping θ_S from $h_X(S)$ to $H(S)$. Applied to $S = X$, one gets ζ as $\theta_X(1_X)$. For any S and $g\colon S \to X$, $\theta_S(g) = \theta_S(h_X(g)(1_X)) = H(g)(\theta_X(1_X)) = H(g)(\zeta)$, so θ is determined by ζ.

B.18. Both parts follow readily from the exchange property.

B.19. By (1) of the preceding exercise, $(\theta * 1_h) \circ (1_{1_X} * \theta) = (1_h * \theta) \circ (\theta * 1_{1_X})$. Since $1_{1_X} * \theta = \theta * 1_{1_X} = \theta$ is invertible, the required equation follows.

B.25. Given α from (φ, Φ) to (ψ, Ψ) and β from (ψ, Ψ) to (ω, Ω), define $\beta \circ \alpha\colon U' \to R$ to be the composite

$$U' \xrightarrow{(\alpha,\beta)} R \;{}_t{\times}_s\; R \xrightarrow{m} R.$$

Given (φ', Φ') and (ψ', Ψ') from $R'' \rightrightarrows U''$ to $R' \rightrightarrows U'$, and a 2-morphism β from (φ', Φ') to (ψ', Ψ'), define $\alpha * \beta\colon U'' \to R$ to be the composite

$$U'' \xrightarrow{(\Phi\beta,\alpha\psi')} R \;{}_t{\times}_s\; R \xrightarrow{m} R$$

(which is equal to $m \circ (\alpha\varphi', \Psi\beta)$).

B.30. In (3), the fact that σ^S is a natural transformation follows from the exchange property: given $\rho\colon h \Rightarrow h'$, $(1_g * \rho) \circ (\sigma * 1_h) = (\sigma * 1_{h'}) \circ (1_f * \rho)$.

B.31. That (1) implies (5) and (7) implies (2) are similar to the proofs that (1) implies (3) and (4) implies (2). To see that (7) implies (1), the essential surjectivity of f^Y, applied to 1_Y, provides a $g\colon Y \to X$ and a 2-isomorphism $f \circ g \overset{\sim}{\Rightarrow} 1_Y$; f_X essentially surjective, applied to 1_X, provides a $g'\colon Y \to X$ and a 2-isomorphism $f \circ g' \overset{\sim}{\Rightarrow} 1_X$. The equivalence of (2), (8), (9) and (10) is seen by taking η and ψ to be inverses of each other, and taking θ and ϕ to be inverses of each other.

B.34. If $H\colon X \times [0, 1] \to Y$ is a homotopy, define a 2-morphism by the mapping from $\pi(X)_0$ to $\pi(Y)_1$ that sends a point x in X to the path $t \mapsto H(x, t)$ in Y.

B.43. That the diagrams commute follows from several applications of the exchange property, together with the identity $(1_p * \epsilon^p) \circ (\eta^p * 1_p) = 1_p$ for the first diagram, $(\epsilon^q * 1_{q'}) \circ (1_{q'} * \eta^q) = 1_{q'}$ for the second, and $(1_f * \epsilon^f) \circ (\eta^f * 1_f) = 1_f$ for the third. For (2), consider the diagram

$$\begin{array}{ccccccccc}
p' \circ f & = & p' \circ 1_Z \circ f & \xRightarrow{\eta^f} & p' \circ f \circ f' \circ f & \xRightarrow{\epsilon^f} & p' \circ f \circ 1_X & = & p' \circ f \\
\| & & & & \Downarrow c_\alpha & & \Downarrow c_\alpha & & \Downarrow c_\alpha \\
1_Y \circ p' \circ f & \xRightarrow[\eta^g]{} & g \circ g' \circ p' \circ f & \xRightarrow[\alpha']{} & g \circ q' \circ f' \circ f & \xRightarrow[\epsilon^f]{} & g \circ q' \circ 1_X & = & g \circ q'
\end{array}$$

The left rectangle commutes by the commutativity of the first diagram in (1), and the other squares commute by the exchange property. The map along the top is the identity on $p' \circ f$, by the defining property of $(f, f', \eta^f, \epsilon^f)$. The assertions of (3) follow from the other two commutative diagrams of (1).

B.35. If $K\colon X \times [0,1] \times [0,1] \to Y$ gives an equivalence from the homotopy H to H', then θ_K, defined by $\theta_K(\sigma)(t_1, \ldots, t_{n+2}) = K(\sigma(t_3, \ldots, t_{n+2}), t_2, t_1)$ gives an equivalence from α_H to $\alpha_{H'}$. If f, g, and h map X to Y, and H_1 is a homotopy from f to g, and H_2 is a homotopy from g to h, then θ_K defines an equivalence between $\alpha_{H_2} \circ \alpha_{H_1}$ and $\alpha_{H_2 \circ H_1}$, where K is defined by the formula $K(x,s,t) = H_1(x, s+2t)$ if $s + 2t \le 1$, and $K(x,s,t) = H_2(x, (s+2t-1)/(s+1))$ if $s + 2t \ge 1$.

B.44. These are proved with several applications of the exchange property, as well as the identity $(1_q * \epsilon^q) \circ (\eta^q * 1_q) = 1_q$ (for (1)).

B.46. This takes a homomorphism $f\colon G \to H$ to the functor $B(f)\colon BG \to BH$ that takes a (right) G torsor E to the H-torsor $E \times_f H = E \times H / \{(v \cdot x, y) \sim (v, f(x) y)\}$. If also $g\colon H \to K$, $\gamma^B_{f,g}$ takes E to the isomorphism from $E \times_{gf} K$ to $(E \times_f H) \times_g K$ that takes (v,z) to $((v,1),z)$, and δ^B_G takes E to the isomorphism from $E \times_{\mathrm{id}} G$ to E that takes (v,x) to $v \cdot x$. If a in H gives a 2-morphism from f to g, for f and g homomorphisms from G to H, then $B(a)$ is the natural transformation from $B(f)$ to $B(g)$ that takes a G-torsor E to the map $E \times_f H$ to $E \times_g H$, $(v,y) \mapsto (v, a^{-1} y)$.

B.48. Use property (c) for β, together with the exchange property.

B.53. We will carry this out in one typical case, proving the first half of the pseudofunctor property (b) for G, with its associated 2-isomorphisms $\gamma^G_{f,g}$ and δ^G_X. That is, we show that the diagram

$$\begin{array}{ccc}
G(f) & = & G(f) \circ 1_{G(X)} \\
\| & & \Uparrow \delta^G_X \\
G(f \circ 1_X) & \xRightarrow[\gamma^G_{1_X, f}]{} & G(f) \circ G(1_X)
\end{array}$$

commutes. Applying F, one needs the upper left square in the diagram

$$\begin{array}{ccccccc}
FG(f) & = & F(G(f)\circ 1_{G(X)}) & \overset{\gamma^F_{1_{G(X)},G(f)}}{\Longrightarrow} & FG(f)\circ F(1_{G(X)}) & \overset{\delta^F_{G(X)}}{\Longrightarrow} & FG(f)\circ 1_{FG(X)} \\
\Vert & & \Uparrow{\scriptstyle F(1_{G(f)}*\delta^G_X)} & & \Uparrow{\scriptstyle F(\delta^G_X)} & & \Downarrow{\scriptstyle \Theta_X} \\
FG(f\circ 1_X) & \underset{F(\gamma^G_{1_X,f})}{\Longrightarrow} & F(G(f)\circ G(1_X)) & \underset{\gamma^F_{G(1_X),G(f)}}{\Longrightarrow} & FG(f)\circ FG(1_X) & \underset{\lambda_{1_X}}{\Longrightarrow} & FG(f)\circ\beta'_X\circ\beta_X \\
{\scriptstyle\lambda_f}\Downarrow & & & & {\scriptstyle\lambda_f*\lambda_{1_X}}\Downarrow & \swarrow{\scriptstyle\lambda_f} & \\
\beta'_Y\circ f\circ\beta_X & = & \beta'_Y\circ f\circ 1_X\circ\beta_X & \underset{\Theta'_X}{\Longrightarrow} & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X & &
\end{array}$$

to commute. The upper center square commutes by property (d) for F; the upper right square commutes by the definition of γ^G_X; the lower right triangle commutes by the exchange property (e) for 2-categories. So we are reduced to showing that the outside diagram commutes. For this we fill in the diagram differently:

$$\begin{array}{ccccc}
FG(f) & = \quad F(G(f)\circ 1_{G(X)}) \quad \overset{\gamma^F_{1_{G(X)},G(f)}}{\Longrightarrow} & FG(f)\circ F(1_{G(X)}) & & \\
\Vert & & \Downarrow{\scriptstyle\delta^F_{G(X)}} & & \\
FG(f) & = & FG(f)\circ 1_{FG(X)} & \overset{\Theta_X}{\Longrightarrow} & FG(f)\circ\beta'_X\circ\beta_X \\
{\scriptstyle\lambda_f}\Downarrow & & {\scriptstyle\lambda_f}\Downarrow & & \Downarrow{\scriptstyle\lambda_f} \\
\beta'_Y\circ f\circ\beta_X & = & \beta'_Y\circ f\circ\beta_X\circ 1_{FG(X)} & \underset{\Theta_X}{\Longrightarrow} & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X \\
\Vert & & & & \Vert \\
\beta'_Y\circ f\circ 1_X\circ\beta_X & & \underset{\Theta'_X}{\Longrightarrow} & & \beta'_Y\circ f\circ\beta_X\circ\beta'_X\circ\beta_X
\end{array}$$

The top square commutes by property (b) for F; the next square down commutes by property (c) of a 2-category; the one to its right commutes by the exchange property (e). Finally, the lower rectangle commutes by the key property that

$$\begin{array}{ccc}
\beta_X & = & \beta_X\circ 1_{FG(X)} \\
\Vert & & \Downarrow{\scriptstyle\Theta_X} \\
1_X\circ\beta_X & \underset{\Theta'_X}{\Longrightarrow} & \beta_X\circ\beta'_X\circ\beta_X
\end{array}$$

commutes. Most of the other verifications are similar to this.