CHAPTER 2

Categories fibered in groupoids

This chapter sets up the first structures which will play a role in the theory of stacks. There is a base category, which for us will usually be a category of schemes. Over the base category we will consider categories where generally an object consists of an object of the base category plus some extra structure. Usually we are motivated by a moduli problem, so we could be considering a scheme S together with a geometric object such as a family of curves, on S.

We will start off by providing a host of examples of such categories to provide insight into the abstract definitions and constructions that follow. Most of these examples will end up being algebraic stacks. One feature that we will be able to observe immediately, however, is that we are always looking at objects or structures that can be pulled back along an arbitrary morphism of schemes. This statement is formalized by the notion of *fibered category*. Actually it is more important for the theory of stacks to consider a somewhat stronger notion, that of *categories fibered in groupoids*. For this, two basic axioms detailed in §2.3 assert that pullbacks of objects exist, up to a canonical isomorphism, and that these objects themselves are allowed to have additional automorphisms. It is this latter feature that makes *CFGs* (categories fibered in groupoids) well suited for the study of moduli problems. In the chapters that follow we will be developing the extra conditions to be satisfied for a CFG to be a stack, and eventually for a stack to be an algebraic stack.

1. The base category $\mathcal{S}$

We have seen that stacks are defined over a base category $\mathcal{S}$. Usually this will be a category of schemes, either all schemes (Sch) or schemes (Sch/Λ) over a fixed base scheme Λ. Often we take $\Lambda = \mathrm{Spec}(k)$, for k a field, or more generally a (commutative) base ring. This may be restricted to a smaller category, say schemes of finite type over k. For example, $\mathcal{S}$ may be taken to be the category of quasi-projective schemes over the complex numbers. For technical reasons, it is sometimes convenient to allow schemes that are merely locally of finite type over k. It is important, however, that $\mathcal{S}$ be closed under formation of arbitrary fiber products $X \times_Z Y$. In particular, one cannot limit oneself to reduced, irreducible varieties; nilpotent elements in the structure sheaves must be allowed. We write $X \times Y$ for $X \times_\Lambda Y$. All schemes will be understood to be in $\mathcal{S}$ unless otherwise stated. (See the Glossary for some basic notions about schemes and morphisms.)

We will sometimes abuse language by regarding Λ as a point, even if Λ is not Spec of a field. For example, we will say that G is an algebraic group instead of saying that G is a group scheme over Λ.

It can be useful, especially in a first reading, to take $\mathcal{S}$ to be the category of analytic spaces, where constructions are often easier, or even the category of topological spaces. Another variation is important, and is that taken in [**61**]: $\mathcal{S}$ can be the category of affine schemes. This is consistent with the point of view that any scheme or scheme-like object should be constructible from affine schemes, or that a scheme is determined by knowing all morphisms of affine schemes into it. We will not take this track, however, at least at this stage. (With this variation, one has to distinguish between schemes in $\mathcal{S}$ and general schemes.)

The base category $\mathcal{S}$ will come with a Grothendieck topology. For us this will mean that we have a notion of a **covering** of a scheme S, which is a collection of morphisms $\{U_\alpha \to S\}$, such that each point of S is in the image of at least one of these maps. (See the Glossary for precise definition.) The topologies that we may consider are: (1) the Zariski topology, where the $U_\alpha \to S$ are open embeddings; (2) the étale topology, where each of the maps is étale; (3) the smooth topology, where each map is smooth; (4) the flat topology, where each map is flat and locally of finite presentation. The relevant topology will generally be the étale topology; we will eventually see results that say we can just as well use the smooth or flat topology. (The Zariski topology is used only in examples.) A single map $U \to S$ is called a **covering map** if $\{U \to S\}$ is a covering. Any covering $\{U_\alpha \to S\}$ determines a covering map $U = \coprod U_\alpha \to S$, which can often be used in place of the covering. When we have a notion of triviality, we will say that something is trivial in the étale topology when its pullback to each U_α in such a covering is trivial; this will be equivalent to the single pullback to $U = \coprod U_\alpha$ being trivial.

In Chapters 2 and 3, in fact, the topology on $\mathcal{S}$ will be used only in some examples, for which one needs a notion of "locally trivial" in some topology. The general discussion here makes sense when $\mathcal{S}$ is any category with fiber products. The topology will come into play in a serious way in Chapter 4 when we state the definition of a stack. Only when the final axioms for a Deligne–Mumford stack are introduced in Chapter 5 will the fact that $\mathcal{S}$ is a category of schemes be used.

2. Examples

A stack is not any kind of space with some structure; rather, it is a *category*. A stack (over $\mathcal{S}$) is a category $\mathfrak{X}$ together with a functor $p\colon \mathfrak{X} \to \mathcal{S}$, satisfying some properties. A category together with a functor to another category, with an appropriate notion of pullbacks, is known as a **fibered category**. Our fibered categories will all be fibered over $\mathcal{S}$. First we look at some examples of such categories — many of which will turn out to be stacks, at least with appropriate added hypotheses (such as a condition to be locally trivial, or some stability condition). We will describe the objects and morphisms in the category $\mathfrak{X}$. Usually the compositions of morphisms will be obvious. The easy verifications that $\mathfrak{X}$ is a category, and p a functor, are left to the reader.

Example 2.1. Let X be a scheme (understood to be in $\mathcal{S}$). Then X determines a stack, which we will denote for now by $\underline{X}$. An object in the category $\underline{X}$ is a scheme S together with a morphism $f\colon S \to X$. A morphism from the object $f'\colon S' \to X$ to the object $f\colon S \to X$ is a morphism of schemes $g\colon S' \to S$ such that $f \circ g = f'$. Composites are defined in the obvious way. The functor $p\colon \underline{X} \to \mathcal{S}$ takes the object $f\colon S \to X$ to the scheme S; a morphism from $f'\colon S' \to X$ to $f\colon S \to X$ is taken to the corresponding morphism from S' to S.

One case deserves special mention. Let us consider $\mathcal{S} = (\mathrm{Sch}/\Lambda)$. When $X = \Lambda$, then $\underline{X}$ is the category $\mathcal{S}$ itself, and p is the identity functor. For then an object is a scheme S over Λ, i.e., equipped with a structure map $S \to \Lambda$, together with a morphism $f\colon S \to \Lambda$. This has to be a morphism over Λ, and that means that f must be equal to the structure map of S. In other words, an object of $\underline{\Lambda}$ is a scheme S with its structure map to Λ, i.e., an object of $\mathcal{S}$.

Example 2.2. For a nonnegative integer g, there is a category $\mathcal{M}_g$, the moduli stack of curves of genus g. The objects of $\mathcal{M}_g$ are smooth projective morphisms $\pi\colon C \to S$, whose geometric fibers are connected curves of genus g. A morphism from $\pi'\colon C' \to S'$ to $\pi\colon C \to S$ is a morphism from C' to C and a morphism from S' to S making a cartesian diagram with π' and π. If a fiber product $C_{S'} = C \times_S S'$ is fixed, this is the same as giving an isomorphism of $C' \to S'$ with $C_{S'} \to S'$. The map from $\mathcal{M}_g$ to $\mathcal{S}$ takes the family $\pi\colon C \to S$ to S, and a morphism to the constituent map $S' \to S$. Composites are defined in the evident way:

$$\begin{tikzcd} C'' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d] & C \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

noting that the outer diagram is cartesian if each of the inner diagrams is cartesian.

Example 2.3. Let G be an algebraic group (i.e., a group scheme over Λ). This defines a category BG, whose objects are principal G-bundles. A **principal G-bundle**, or **torsor**, is a pair of schemes S and E with a morphism from E to S and a right action $E \times G \to E$ of G on E. The trivial G-torsor over S is that with $E = S \times G$, with the right action of G on the second factor G only, and $E \to S$ the first projection. If $f\colon T \to S$ is any morphism, we have a pullback f^*E over T. This is defined by $f^*E = T \times_S E$, with induced map to T and induced action of G. We require that a G-torsor be locally trivial in the given topology on S. This means that there exists a covering map $f\colon T \to S$ such that the pullback f^*E is trivial (isomorphic to the trivial G-torsor on T). We will usually work with the étale topology, meaning f should be étale and surjective.

The category BG has these G-torsors as its objects. A morphism from a G-torsor $E' \to S'$ to a G-torsor $E \to S$ is given by a morphism $S' \to S$ and a G-equivariant morphism $E' \to E$ such that the diagram

$$\begin{tikzcd} E' \arrow[r] \arrow[d] & E \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

is cartesian. As in Example 2.2, if a pullback of $E \to S$ by $S' \to S$ is fixed, this is the same as specifying an isomorphism of $E' \to S'$ with this pullback. Compositions and the mapping to $\mathcal{S}$ are defined as in Example 2.2.

EXAMPLE 2.4. Let $h\colon \mathcal{S} \to (\mathrm{Set})$ be any contravariant functor from our category of schemes to the category of sets. This determines a category which we denote for now by $\underline{h}$. The objects of $\underline{h}$ are pairs (S, α), with S a scheme in $\mathcal{S}$ and α an element of the set $h(S)$. A morphism from (S', α') to (S, α) is a map $\varphi\colon S' \to S$ in $\mathcal{S}$ such that $h(\varphi)\colon h(S) \to h(S')$ maps α to α'. The projection from $\underline{h}$ to $\mathcal{S}$ takes (S, α) to S.

Example 2.1 is a special case of Example 2.4. In fact, a scheme X determines a contravariant functor h_X from $\mathcal{S}$ to (Set), the functor of points $h_X(S) = \mathrm{Hom}_{\mathcal{S}}(S, X)$. Then $\underline{X}$ is the category $\underline{h_X}$.

Many of the stacks that are met in practice are variations of these four examples. Here are a few of these:

EXAMPLE 2.5. There is a category $\mathcal{M}_{g,n}$ of n-pointed curves of genus g. Its objects are smooth projective morphisms $\pi\colon C \to S$, whose geometric fibers are connected curves of genus g, together with disjoint sections $\sigma_1, \dots, \sigma_n$. (These sections are morphisms $\sigma_i\colon S \to C$ such that $\pi \circ \sigma_i = \mathrm{id}_S$, which give n distinct points in each geometric fiber.) Morphisms are defined as in Example 2.2, with the added requirement that the sections of the first family are mapped to the sections of the second. The projection to $\mathcal{S}$ is defined as in Example 2.2.

Recall, an elliptic curve is a curve of genus 1 together with a chosen point (the identity element for the group structure). Then $\mathcal{M}_{1,1}$ is the category of elliptic curves.

EXAMPLE 2.6. Suppose an algebraic group G acts (on the right) on a scheme X. There is a category denoted $[X/G]$, whose objects are G-torsors $E \to S$ (with action $E \times G \to E$), together with an equivariant morphism from E to X. Morphisms are defined as in Example 2.3, with the additional condition that the map from E' to E must form a commutative triangle with the maps to X. The functor $p\colon [X/G] \to \mathcal{S}$ again maps $(E \to S,\ E \to X)$ to S. Note that when $X = \Lambda$, we recover the category BG, i.e., $[\Lambda/G] = BG$.

EXAMPLE 2.7. For a positive integer n, let $\mathcal{V}_n$ be the category of vector bundles of rank n. The objects are vector bundles $E \to S$, and the morphisms from $(E' \to S')$ to $(E \to S)$ are given by a cartesian diagram as in Example 2.3, that identifies $E' \to S'$ via a bundle isomorphism with a pullback bundle $E \times_S S' \to S'$. The functor to $\mathcal{S}$ takes $E \to S$ to S.

Example 2.8. For a positive integer n, let $\mathcal{C}_n$ be the category of covering spaces of degree n. An object is a finite étale morphism $X \to S$ of degree n, and a morphism from $X' \to S'$ to $X \to S$ is again a cartesian diagram.

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be categories over $\mathcal{S}$. A **morphism** from $\mathfrak{X}$ to $\mathfrak{Y}$ is a functor $f\colon \mathfrak{X} \to \mathfrak{Y}$ commuting with the given functors to $\mathcal{S}$. Functors between two categories $\mathfrak{X}$ and $\mathfrak{Y}$ do not necessarily form a set; rather, they form a category. The objects are functors from $\mathfrak{X}$ to $\mathfrak{Y}$, and the morphisms are natural transformations between functors; recall that a natural transformation from f_1 to f_2 assigns to each object x in F a morphism from $f_1(x)$ to $f_2(x)$ in G, which is compatible with morphisms (see the Glossary). Given categories $\mathfrak{X}$ and $\mathfrak{Y}$, and projection functors $p\colon \mathfrak{X} \to \mathcal{S}$ and $q\colon \mathfrak{Y} \to \mathcal{S}$, we denote by $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$ the following category. The objects are functors $f\colon \mathfrak{X} \to \mathfrak{Y}$ satisfying $q \circ f = p$. The morphisms from f_1 to f_2 are natural isomorphisms from f_1 to f_2 such that, for all objects x in F, the isomorphism from $f_1(x)$ to $f_2(x)$ maps (via q) to the identity map in $\mathcal{S}$ from $p(x) = q(f_1(x))$ to $p(x) = q(f_2(x))$. A morphism $f\colon \mathfrak{X} \to \mathfrak{Y}$ of categories over $\mathcal{S}$ will be called an **isomorphism** if it is an equivalence of categories.

Example 2.9. Here are some examples of morphisms:

(1) A morphism $f\colon X \to Y$ of schemes determines a functor $\underline{f}\colon \underline{X} \to \underline{Y}$, that takes a scheme $S \to X$ over X to the composite $S \to X \to Y$. Conversely, if $\varphi\colon \underline{X} \to \underline{Y}$ is a functor over $\mathcal{S}$, applying φ to the identity map $X \to X$ (an object in $\underline{X}$), gives a map $f\colon X \to Y$ (the image object in $\underline{Y}$), and one verifies easily that $\varphi = \underline{f}$. In other words,
$$\mathrm{HOM}(\underline{X}, \underline{Y}) = \mathrm{Hom}_{\mathcal{S}}(X, Y).$$
(This means that the category on the left is just a set, meaning it has no maps besides identity maps.)

(2) A homomorphism $\varphi\colon G \to G'$ of algebraic groups determines a functor $BG \to BG'$ that takes a G-torsor $\pi\colon E \to S$ to the G'-torsor
$$E_{G'} = E \times^G G' = E \times G'/\{(x, \varphi(g)g') \sim (x \cdot g, g')\},^1$$
assuming that these quotient schemes exist (see Remark 2.17, below). This is given a right action of G' by $(x, g') \cdot h' = (x, g'h')$, and projection $E_{G'} \to S$ by $(x, g') \mapsto \pi(x)$. Note that if the pullback of $E \to S$ by a map $T \to S$ is isomorphic to the trivial bundle $T \times G \to T$, then the pullback of $E_{G'} \to S$ by the same map is isomorphic to the trivial bundle $T \times G' \to T$. In the familiar setting where $\Lambda = \operatorname{Spec} k$ for k a field, G an algebraic group over k, and $E \to S$ described by means of a covering $S' \to S$ and cocycle data $S' \times_S S' \to G$, then this cocycle data composed with φ serves as cocycle data for the G'-torsor $E_{G'}$.

(3) There is a canonical morphism from $\mathcal{V}_n$ to BGL_n that sends a vector bundle to its associated principal bundle of frames. A vector bundle $E \to S$ comes with a (right) GL_n-action. This induces an action on the n-fold product $E \times_S E \cdots \times_S$

[1] Here, as frequently throughout these notes, we use set-theoretic notation to describe various morphisms or compatibilities, trusting that the reader can construct the correct scheme-theoretic morphisms or commutative diagrams.

E (the diagonal action). The associated principal bundle is the open subscheme of $E \times_S E \cdots \times_S E$ of n-tuples of vectors that are linearly independent.

(4) If an algebraic group G acts on a scheme X, there is a canonical morphism from $\underline{X}$ to $[X/G]$. This takes an object $f\colon S \to X$ of X to the object with trivial torsor $S \times G \to S$, with map $S \times G \to X$ given by $(s, g) \mapsto f(s) \cdot g$.

(5) There is a canonical morphism from $\mathcal{M}_{g,n+1}$ to $\mathcal{M}_{g,n}$, that simply forgets the last section, and a morphism from $\mathcal{M}_{g,n}$ to $\mathcal{M}_g$ that forgets all the sections. The morphism from $\mathcal{M}_{g,1}$ to $\mathcal{M}_g$ can be regarded as the universal curve.

(6) There is a morphism from BG to $\mathcal{C}_n$, where $G = \mathfrak{S}_n$ is the symmetric group. This takes a G-torsor $E \to S$ to the covering $\{1, ..., n\} \times^G E \to S$.

Exercise 2.1. Verify that the map $BGL_n \to \mathcal{V}_n$ of (3) is an isomorphism, i.e., an equivalence of categories. Do the same for the map $B\mathfrak{S}_n \to \mathcal{C}_n$ of (6).

The above examples will be the most important for our discussions. However, we indicate next some of the many variations, a few of which will be discussed later. Some of these are related to an important goal in many moduli problems, that of constructing appropriate compactifications. Others are used to "rigidify" a given moduli problem.

Example 2.10. A compactification $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$ $(g \geq 2)$ by stable curves [**20**]. The objects are projective flat morphisms $\pi\colon C \to S$. Each geometric fiber of π must be a connected, reduced curve of arithmetic genus g, with at most nodes (ordinary double points) as singularities. There is a further stability condition, that any irreducible component of a fiber which is a nonsingular curve of genus 0 must meet other components in at least 3 points. This is a category over $\mathcal{S}$. (Eventually we will see that it is a Deligne–Mumford stack, proper over the base scheme.)

Example 2.11. A compactification $\overline{\mathcal{M}}_{g,n}$ of $\mathcal{M}_{g,n}$. The objects are projective flat morphisms $\pi\colon C \to S$, together with n disjoint sections σ_i. Each geometric fiber of π must be a connected curve of arithmetic genus g, with at most nodes as singularities, and the n points picked out in the fiber by the sections must be nonsingular points. There is, again, a stability condition: we must have $2g+n-2 > 0$, and any irreducible component of a geometric fiber of π that is nonsingular of genus 0 must have at least 3 markings, i.e., points at which it meets other components or points given by the sections.

Example 2.12. Consider $\Lambda = \mathrm{Spec}(\mathbb{C})$. Let X be a smooth projective variety, and β a class in the homology group $H_2(X, \mathbb{Z})$. We have a category $\mathcal{M}_{g,n}(X, \beta)$, whose objects consist of smooth projective families of curves $\pi\colon C \to S$, together with n distinct sections σ_i as in Example 2.6, together with a morphism $\mu\colon C \to X$ with the property that, for each closed point s in S, the induced map $\mu_s\colon C_s = \pi^{-1}(s) \to X$ maps the fundamental class of the curve C_s to β, i.e., $\mu_*[C_s] = \beta$.

Example 2.13. An important tool in quantum cohomology is Kontsevich's compactification $\overline{\mathcal{M}}_{g,n}(X, \beta)$ of $\mathcal{M}_{g,n},(X, \beta)$. The objects are $\pi\colon C \to S$ with σ_i as in Example 2.11, and $\mu\colon C \to X$ as in Example 2.12. In this case the stability condition is that any component that is nonsingular of genus 0 and is mapped to a point by μ

must have three markings among the points where it meets other components or the points given by the n sections.

EXAMPLE 2.14. Consider the category $\underline{\mathrm{Hilb}}_{g,r}$, whose objects consist of projective families $C \to S$ of curves as in Example 2.2 together with $N = (2r-1)(g-1)$ generating sections of the line bundle $\omega_{C/S}^{\otimes r}$, such that the induced map from C to $\mathbb{P}_S^{N-1}$ is a closed embedding; these are defined up to multiplication by scalars as in the preceding example. Here $\omega_{C/S}$ is the relative canonical line bundle, or dualizing sheaf; it is the sheaf of relative differentials $\Omega^1_{C/S}$ if C is smooth over S. We usually assume $g \geq 2$ and $r \geq 2$.

EXAMPLE 2.15. Consider smooth families of curves $\pi\colon C \to S$ over with a **Jacobi level** n **structure**. This is an isomorphism of each $H^1(C_s; \mathbb{Z}/n\mathbb{Z})$ with $(\mathbb{Z}/n\mathbb{Z})^{2g}$, that is symplectic, i.e., takes the cup product pairing (with values in $H^2(C_s; \mathbb{Z}/n\mathbb{Z}) = \mathbb{Z}/n\mathbb{Z}$) to the canonical symplectic pairing on $(\mathbb{Z}/n\mathbb{Z})^{2g}$; these isomorphisms must vary nicely in the family, which means that they are given by a symplectic isomorphism of $R^1\pi_*(\mathbb{Z}/n\mathbb{Z})$ with the trivial sheaf $(\mathbb{Z}/n\mathbb{Z})^{2g}$.

EXAMPLE 2.16. There is a category Qcoh, where an object of Qcoh over S in $\mathcal{S}$ is a quasicoherent sheaf $\mathcal{E}$ on S. A morphism from $\mathcal{E}'$ on S' to $\mathcal{E}$ on S over $f\colon S' \to S$ is a morphism of sheaves $\mathcal{E} \to f_*\mathcal{E}'$ on S such that the corresponding morphism $f^*\mathcal{E} \to \mathcal{E}'$ of sheaves on S' determined by adjunction is an isomorphism.

REMARK 2.17. We explain why the definition of morphism in Example 2.16 is phrased in terms of a morphism of sheaves $\mathcal{E} \to f_*\mathcal{E}'$, and not directly by means of an isomorphism isomorphism of sheaves $f^*\mathcal{E} \to \mathcal{E}'$ of sheaves on S'. The reason is that $f_*\mathcal{E}'$ is well-defined as a sheaf, while the pullback $f^*\mathcal{E}$ is only defined up to (canonical) isomorphism. It is important to be precise about what consistutes a morphism between two objects in any category; in this instance, the most convenient formulation is by means of the push-forward sheaf.

It happens quite frequently that an object of a category is defined only up to canonical isomorphism. This is the case, for instance, with fiber products in the category of schemes. It is also the case with some of the objects of HOM categories in Example 2.9. We now summarize the "fine print" concerning these examples.

In (2), the existence of the quotient scheme $E_{G'}$ is an honest mathematical requirement. It is satisfied when G and G' are affine group schemes (over the base Λ). Then the construction of $E_{G'}$, which can be achieved using descent (see Appendix A), will use the fact that $E \to S$ admits a local trivialization, making $E_{G'}$ locally a product with G'. The quotient $E_{G'}$, when it exists, is defined up to canonical isomorphism; hence (2) describes an object of $\mathrm{HOM}(BG, BG')$ up to canonical isomorphism. In (3) one could, with care, make sense of $E \times_S \cdots \times_S E$ as a well-defined scheme (Spec of a tensor product of sheaves of $\mathcal{O}_S$-algebras) and thereby obtain a particular object of $\mathrm{HOM}(\mathcal{V}_n, BGL_n)$. That this is possible is not important, and it is more natural to view the example as giving an object of $\mathrm{HOM}(\mathcal{V}_n, BGL_n)$ defined up to canonical isomorphism. Examples (4) and (6) describe objects of HOM categories up to canonical isomorphism, because of the use of fiber products and group quotients, respectively.

Example 2.18. Let X be a scheme (over a base Λ), and fix a functor $X \times -$ from (Sch/Λ) to (Sch/Λ). Then one has categories $\mathcal{V}_{X,n}$ and Coh_X, generalizing the example of vector bundles (Example 2.7). An object of $\mathcal{V}_{X,n}$ is a vector bundle of rank n on $X \times S$, with morphisms given by cartesian diagrams as usual. An object of Coh_X is a quasicoherent sheaf $\mathcal{E}$ on $X \times S$ that is finitely presented and flat over S. (This will be a coherent sheaf when X is a finite-type scheme and S is Noetherian, the case that is usually of interest.) A morphism from $\mathcal{E}'$ to $\mathcal{E}$ over $f\colon S' \to S$ is a morphism $\mathcal{E} \to (1_X \times f)_*\mathcal{E}'$ which by adjunction gives an isomorphism $(1_X \times f)^*\mathcal{E} \to \mathcal{E}'$. The functor, sending a vector bundle to its sheaf of sections, is a morphism $\mathcal{V}_{X,n} \to \mathrm{Coh}_X$. There are variants in which one imposes a stability condition.

One can also make stacks out of families of varieties of higher dimension. Important examples are principally polarized abelian varieties, $K3$ surfaces, etc. We will consider a few of these examples later.

3. CFGs over $\mathcal{S}$

The first requirement for a category $\mathfrak{X}$ with $p\colon \mathfrak{X} \to \mathcal{S}$ to be a stack is that it is a **category fibered in groupoids over $\mathcal{S}$**, which we will abbreviate to CFG, or CFG/$\mathcal{S}$. This means that the following two axioms must be satisfied:

Definition 2.19. A *category fibered in groupoids* over a base category $\mathcal{S}$ is a category $\mathfrak{X}$ with functor $p\colon \mathfrak{X} \to \mathcal{S}$ satisfying the following two axioms:

(1) For every morphism $f\colon T \to S$ in $\mathcal{S}$, and object s in $\mathfrak{X}$ with $p(s) = S$, there is an object t in $\mathfrak{X}$, with $p(t) = T$, and a morphism $\varphi\colon t \to s$ in $\mathfrak{X}$ such that $p(\varphi) = f$.

(2) Given a commutative diagram in $\mathcal{S}$

$$\begin{tikzcd} U \arrow[dd, "g"'] \arrow[dr, "h"] & \\ & S \\ T \arrow[ur, "f"'] & \end{tikzcd}$$

with $\varphi\colon t \to s$ in $\mathfrak{X}$ mapping to $f\colon T \to S$, and $\eta\colon u \to s$ in $\mathfrak{X}$ mapping to $h\colon U \to S$, there is a unique morphism $\gamma\colon u \to t$ in $\mathfrak{X}$ mapping to $g\colon U \to T$ such that $\eta = \varphi \circ \gamma$:

$$\begin{tikzcd} u \arrow[dd, dashed, "\gamma"'] \arrow[dr, "\eta"] & \\ & s \\ t \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Axiom (2), applied with $U = T$, $h = f$, and $g = 1_T$, implies that the object t with $\varphi\colon t \to s$ guaranteed by the first axiom is determined up to canonical isomorphism. So Axiom (1) can be regarded as saying that pullbacks of objects exist, and Axiom (2) then tells us that these pullbacks are unique up to canonical isomorphism.

In the example $\mathcal{M}_g$, the pullback of a family $C \to S$ by $T \to S$ is the fibered product $C_T = T \times_S C \to T$ (which is unique up to canonical isomorphism). The verification of Axiom (2) comes down to the universality property of the fibered product. Given a diagram

$$\begin{tikzcd} C_U \arrow[r, dashed] \arrow[d] \arrow[rr, bend left] & C_T \arrow[r] \arrow[d] & C \arrow[d] \\ U \arrow[r] & T \arrow[r] & S \end{tikzcd}$$

with cartesian right-hand square and cartesian outer square, there is a unique dashed arrow making the left-hand square commute and making the curved arrow the composite of the top two horizontal arrows. Notice that the left-hand square must then be cartesian.

We leave it to the reader to verify, using similar reasoning, that these Axioms (1) and (2) are satisfied in each of the other examples that we have seen so far.

Exercise 2.2. Show that Axioms (1) and (2) are equivalent to (1) and

(2′) For every morphism $f\colon T \to S$ in $\mathcal{S}$, and morphisms $\varphi\colon t \to s$ and $\varphi'\colon t' \to s$ in F with $p(\varphi) = f = p(\varphi')$, there is a unique morphism $\vartheta\colon t \to t'$ in F over 1_T such that $\varphi = \varphi' \circ \vartheta$.

For an object S in $\mathcal{S}$, we denote by $\mathfrak{X}_S$ the subcategory of $\mathfrak{X}$ whose objects map to S, and whose morphisms map to the identity map 1_S. It follows from Axiom (2) that every morphism in $\mathfrak{X}_S$ is an isomorphism. (Given a morphism $\varphi\colon t \to s$ in F_S, take $u = s$, and $\eta = 1_s$ to get an inverse γ for φ.) Recall that a groupoid is a category in which every morphism is an isomorphism. This explains the terminology *category fibered in groupoids*: it follows from the axioms that the category $\mathfrak{X}_S$ is a groupoid. When two more axioms are satisfied, to be given in Chapter 4, a CFG qualifies as a full-fledged stack — which gives it the right to discard the awkward name "category fibered in groupoids." Still more will be required for a stack to be an algebraic stack, of either Deligne–Mumford or Artin type.

The next result provides the link between two notions of "S-valued points" of a stack. First, we have the fiber $\mathfrak{X}_S$ just introduced. In a moduli problem, where $\mathfrak{X}$ is a category of families of geometric objects, then $\mathfrak{X}_S$ will be the category of objects over S. But just as for schemes with its functor of points, we can consider the fibered category $\underline{S}$ and look at the category $\mathrm{HOM}(\underline{S}, \mathfrak{X})$. When $\mathfrak{X}$ is a category fibered in groupoids, these two notions are equivalent.

Proposition 2.20. *Let $\mathfrak{X}$ be a category fibered in groupoids over a base category $\mathcal{S}$. Let X be an object of $\mathcal{S}$. Then the functor from $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ to $\mathfrak{X}_X$, given by evaluation at the object $(X, 1_X)$ of $\underline{X}$, is surjective (on objects) and fully faithful. In particular it is an equivalence of categories.*

Proof. We show that the functor is fully faithul and essentially surjective. In fact, the functor is surjective. Given an object x of $\mathfrak{X}_X$, we apply Axiom (1) to every object $(S, f\colon S \to X)$ of $\underline{X}$ to obtain an associated object $x_{(S,f)}$ of $\mathfrak{X}_S$ and morphism $x_{(S,f)} \to x$ over $S \to X$. For X and the identity morphism 1_X we choose $x_{(X,1_X)} = x$. Whenever

$(T, g\colon T \to X)$ is another object of $\underline{X}$, with morphism $T \to S$ in $\underline{X}$, we obtain from Axiom (2) a unique morphism $x_{(T,g)} \to x_{(S,f)}$ making a commutative triangle with the morphisms $x_{(T,g)} \to x$ and $x_{(S,f)} \to x$. The association of the object $x_{(S,f)}$ to (S, f) and the morphism $x_{(T,g)} \to x_{(S,f)}$ to $T \to S$, is a functor from $\underline{X}$ to $\mathfrak{X}$; the verification of this uses the uniqueness assertion of Axiom (2). The functor is an object of $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ which, when evaluated at $(X, 1_X)$, produces x.

To see that the functor is fully faithful, consider a pair of objects (functors) h and h' in $\mathrm{HOM}(\underline{X}, \mathfrak{X})$. To give a morphism in $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ from h to h' is to give a morphism $h(S, f) \to h'(S, f)$ in $\mathfrak{X}_S$ for every object $(S, f\colon S \to X)$ of $\underline{X}$, such that for any object $(T, g\colon T \to X)$ in $\underline{X}$ and morphism $T \to S \to X$ in $\underline{X}$ the square

$$\begin{tikzcd} h(T,g) \arrow[r] \arrow[d] & h(S,f) \arrow[d] \\ h'(T,g) \arrow[r] & h'(S,f) \end{tikzcd}$$

commutes.

Set $x = h(X, 1_X)$ and $x' = h'(X, 1_X)$. Suppose that two morphisms α, $\beta\colon h \Rightarrow h'$ in the category $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ yield the same morphism $\varphi\colon x \to x'$ when evaluated at $(X, 1_X)$. Then the commutative square for $S \to X \to X$, where the second map is 1_X, yields

$$\begin{tikzcd} h(S,f) \arrow[r] \arrow[d] & x \arrow[d, "\varphi"] \\ h'(S,f) \arrow[r] & x' \end{tikzcd}$$

where the left-hand map is $\alpha(S, f)$ or $\beta(S, f)$. By the uniqueness assertion of Axiom (2), we have $\alpha(S, f) = \beta(S, f)$. Now let $\varphi\colon x \to x'$ be an arbitrary morphism in $\mathfrak{X}_X$; we need to exhibit a morphism $\alpha\colon h \Rightarrow h'$ in the category $\mathrm{HOM}(\underline{X}, \mathfrak{X})$ which, when evaluted at $(X, 1_X)$, produces φ. Given an object (S, f) of $\underline{X}$, we define $\alpha(S, f)\colon h(S, f) \to h'(S, f)$, using Axiom (2), to be the unique morphism over 1_S whose composite with $h'(S, f) \to x'$ is equal to the composite $h(S, f) \to x \to x'$. Now given $T \to S$ in $\underline{X}$, we have a diagram

$$\begin{tikzcd} h(T,g) \arrow[r] \arrow[d, "\alpha(T,g)"'] & h(S,f) \arrow[r] \arrow[d, "\alpha(S,f)"'] & x \arrow[d, "\varphi"] \\ h'(T,g) \arrow[r] & h'(S,f) \arrow[r] & x' \end{tikzcd}$$

where the right-hand square and outer square commute. Again by Axiom (2) it follows that the left-hand square commutes. □

If x is an object of $\mathfrak{X}$ over X, then we will frequently use the same symbol x to denote a morphism $\underline{X} \to \mathfrak{X}$ which yields x when evaluated at $(X, 1_X)$. So, for instance, if $\mathfrak{X} = BG$ and $E \to X$ is a G-torsor, then we will have $E\colon \underline{X} \to BG$. This morphism (functor between categories) is determined up to a canonical natural isomorphism of functors).

The proof of Proposition 2.20 makes heavy use of the existence of choices of pullbacks of a given object of $\mathfrak{X}$. It is convenient to formalize the existence of pullbacks in the form of a pullback (or change of base) functor. Let $f\colon T \to S$ be a morphism in $\mathcal{S}$. For every object s in $\mathfrak{X}_S$, fix an object t in $\mathfrak{X}_T$ with $t \to s$ as in Axiom (1); we then use the common notation $f^*(s)$, or sometimes s_T or $s|_T$, for this object t, and call it the pullback. If we have a morphism $\varphi\colon s' \to s$ in $\mathfrak{X}_S$, it follows from Axiom (2) that there is a unique morphism ψ from $f^*(s')$ to $f^*(s)$ in $\mathfrak{X}_T$ such that the diagram

$$\begin{tikzcd} f^*(s') \arrow[r] \arrow[d, "\psi"'] & s' \arrow[d, "\varphi"] \\ f^*(s) \arrow[r] & s \end{tikzcd}$$

commutes; this map ψ is denoted $f^*(\varphi)$. These choices determine a functor f^* from $\mathfrak{X}_S$ to $\mathfrak{X}_T$, called the **change of base functor**. If $f\colon S \to S$ is the identify morphism, we choose f^* to be the identity. If $f\colon T \to S$ and $g\colon U \to T$, there is a canonical natural isomorphism of functors $g^* \circ f^* \cong (f \circ g)^*$ from $\mathfrak{X}_S$ to $\mathfrak{X}_U$. In addition, these isomorphisms satisfy the expected "cocycle" compatibility condition with a third map $h\colon V \to U$, specifically that the diagram

$$\begin{array}{ccccc} h^* \circ (g^* \circ f^*) & \cong & h^* \circ (f \circ g)^* & \cong & ((f \circ g) \circ h)^* \\ \| & & & & \| \\ (h^* \circ g^*) \circ f^* & \cong & (g \circ h)^* \circ f^* & \cong & (f \circ (g \circ h))^* \end{array}$$

commutes.

It is often simplest to think of $\mathfrak{X}$ as the collection of groupoids $\mathfrak{X}_S$, together with the pullbacks $f^*\colon \mathfrak{X}_S \to \mathfrak{X}_T$ for morphisms $f\colon T \to S$. This has all the essential information. By using the original definition as one category with a functor to $\mathcal{S}$, however, one avoids having to verify these cocycle conditions. Note that in Example 2.4 (and therefore also in Example 2.1), the only morphisms in the categories $\mathfrak{X}_S$ are the identity morphisms; it is the presence of nontrivial automorphisms in the other examples that make general stacks richer than ordinary schemes or functors to sets.

Remark 2.21. In the previous section we introduced the first of a series of axioms for a category to be a stack, namely that it should be a category fibered in groupoids over the base category. A more general notion, that of being a *fibered category*, appears in the literature; the difference is that the fibers are allowed to be arbitrary categories, rather than categories whose morphisms are all isomorphisms (groupoids). We outline the differences between the two notions here.

In a fibered category, Axiom (2) of Definition 2.19 is not required to hold for all diagrams and given morphisms. Rather, a morphism $\varphi\colon t \to s$ over $f\colon T \to S$ is defined to be *cartesian* if, for every U and u, the conclusion of Axiom (2) holds. Then, a fibered category is defined as a category $\mathfrak{X}$ with a functor to a base category $\mathcal{S}$, such that for every morphism $f\colon T \to S$ in $\mathcal{S}$ and object s in $\mathfrak{X}$ over S, there exists an object t in $\mathfrak{X}$ over T and a cartesian morphism $\varphi\colon t \to s$ in $\mathfrak{X}$ over f.

A CFG is then a fibered category in which all the morphisms are cartesian. Some of our examples of CFG sit inside larger fibered categories. In the stack of elliptic curves $\mathcal{M}_{1,1}$, for instance, we may allow arbitrary commutative diagrams $C' \to C$, $S' \to S$ as morphisms, rather than only cartesian diagrams. This produces a bigger category, which is a fibered category (this bigger category won't be an algebraic stack). Similarly, the categories $\mathcal{V}_n$, Qcoh, $\mathcal{V}_{X,n}$, and Coh_X sits inside larger categories, where we allow arbitrary morphisms, not only morphisms that identify a sheaf over S' with the pull-back of a sheaf over S.

Algebraic stacks are always categories fibered in groupoids, so we will not have much use for more general fibered categories. However we point out that the theory of descent (Appendix A) could be stated in the language of fibered categories. For instance, Theorem A.2 could be abbreviated to the statement that the fibered category of quasi-coherent sheaves over schemes is a stack (for the fpqc or fppf topology on schemes). In this book, we choose instead to present the results in Appendix A in an explicit manner, and we make the convention that stacks will be categories fibered in groupoids satisfying additional hypotheses.

An important remark is that a morphism of CFGs $f\colon \mathfrak{X} \to \mathfrak{Y}$ is an isomorphism (equivalence of categories) if and only if there exists a morphism in the other direction $g\colon \mathfrak{Y} \to \mathfrak{X}$, together with 2-isomorphisms $g \circ f \Rightarrow 1_{\mathfrak{X}}$ and $f \circ g \Rightarrow 1_{\mathfrak{Y}}$. Indeed, there is the familiar statement, stated as Proposition B.1, that a functor between categories is an equivalence of categories if and only if it is fully faithful and essentially surjective. Following the proof of Proposition B.1, we need to assign to each object t of $\mathcal{Y}$ (over some T in $\mathcal{S}$) an object $g(t)$ of $\mathcal{X}$. By essential surjectivity there exists an object $\tilde{t}$ of $\mathcal{X}$ and an isomorphism $f(\tilde{t}) \to t$; the isomorphism will be over some isomorphism $\varphi\colon \widetilde{T} \to T$, possibly not the identity. But then we define $g(t)$ to be $(\varphi^{-1})^*\tilde{t}$ and have an isomorphism $f(g(t)) \to t$ over 1_T. The rest of the verification can be copied from the proof of Proposition B.1.

4. 2-commutative diagrams

Given CFGs $\mathfrak{X}$ and $\mathfrak{Y}$ over a base category $\mathcal{S}$, we have seen that morphisms of CFGs from $\mathfrak{X}$ to $\mathfrak{Y}$ (which are functors) form a category $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$, with natural isomorphisms of functors as morphisms in $\mathrm{HOM}(\mathfrak{X}, \mathfrak{Y})$. It will come as no surprise, then, that the natural way to compare two morphisms in $\mathfrak{X}$ to $\mathfrak{Y}$ is to say that they are isomorphic. Often the morphisms will be canonically isomorphic. But it happen much more rarely that the morphisms will actually be equal.

This is particularly the case when the morphisms that we are comparing are gotten by composing other morphisms. Most "commutative" diagrams won't actually commute! Rather, they will commute "up to" a natural isomorphism. We give some examples of this.

Example 2.22. Here are some diagrams of CFGs.

(1) Let G, G', and G'' be affine algebraic groups, and let

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & G' \arrow[d] \\ & G'' \end{tikzcd}$$

be a commutative diagram of algebraic group homomorphisms. Then there is a diagram of CFGs

$$\begin{tikzcd} BG \arrow[r] \arrow[dr] & BG' \arrow[d] \\ & BG'' \end{tikzcd}$$

which commutes up to a canonical natural isomorphism. For instance, if $G'' = G$ and the homomorphsm $G \to G'' = G$ is the identity, then the composite $BG \to BG' \to BG$ will not be equal, but only naturally isomorphic, to the identity 1_{BG}.

(2) Consider the pair of morphisms from $\mathcal{M}_{g,2}$ to $\mathcal{M}_{g,1}$ which forget the first, resp. the second section. These fit into a diagram

$$\begin{tikzcd} \mathcal{M}_{g,2} \arrow[r] \arrow[d] & \mathcal{M}_{g,1} \arrow[d] \\ \mathcal{M}_{g,1} \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

an honest example of a diagram that actually commutes! However, there is a similar operation on n-pointed *stable* curves of genus g (Example 2.11), which forgets one of the sections σ_i of $\pi\colon C \to S$ and collapses any components of the fibers of π which are thereby made unstable. The corresponding diagram

$$\begin{tikzcd} \overline{\mathcal{M}}_{g,2} \arrow[r] \arrow[d] & \overline{\mathcal{M}}_{g,1} \arrow[d] \\ \overline{\mathcal{M}}_{g,1} \arrow[r] & \overline{\mathcal{M}}_g \end{tikzcd}$$

commutes up to a canonical natural isomorphism. That is, the results of forgetting and stabilizing the two markings in either order are canonically isomorphic.

Definition 2.23. A diagram of CFGs is said to be **2-commutative** if it commutes up to a given isomorphism in the relevant HOM category; it is **strictly commutative** if it commutes exactly. An isomorphism between two objects in $\mathrm{HOM}(\mathfrak{X},\mathfrak{Y})$ is called a **2-morphism**, or **2-isomorphism**. (We recall, $\mathrm{HOM}(\mathfrak{X},\mathfrak{Y})$ is a groupoid, i.e., every morphism in $\mathrm{HOM}(\mathfrak{X},\mathfrak{Y})$ is an isomorphism.)

If f, $g\colon \mathfrak{X} \to \mathfrak{Y}$ are morphisms, a 2-morphism can be denoted by $f \Rightarrow g$. That a diagram is 2-commutative can be indicated by marking it with $\Rightarrow$. So, for instance,

$$\begin{tikzcd} BG \arrow[r] \arrow[dr] & BG' \arrow[d] \arrow[dl, phantom, "\Rightarrow" near start] \\ & BG'' \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \overline{\mathcal{M}}_{g,2} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \overline{\mathcal{M}}_{g,1} \arrow[d] \\ \overline{\mathcal{M}}_{g,1} \arrow[r] & \overline{\mathcal{M}}_{g} \end{tikzcd}$$

In fact, CFGs over $\mathcal{S}$ form a 2-category, a richer structure than just a category. In a 2-category there are objects (in this case, CFGs), morphisms (which, for CFGs, are functors), and 2-morphisms, which for CFGs we have just introduced in Definition 2.23. The formalism of 2-categories is not necessary in these early chapters; the reader who wants to look ahead can turn to Appendix B. For now, the main point is that CFGs are part of a structure that is different from an ordinary category. So, for instance in the next section when we discuss fiber products of CFGs, we cannot just use the standard notion of fiber products in a category; a dedicated discussion of the topic will be required.

Exercise 2.3. Consider $\Lambda = \mathrm{Spec}(k)$ where k is an algebraically closed field. Let G be a finite group.

(i) Consider the morphism $f\colon \underline{\Lambda} \to BG$ which assigns to a scheme S the trivial G-torsor $S \times G$. Show that the automorphism group of f in $\mathrm{HOM}(\underline{\Lambda}, BG)$ can be identified with G.

(ii) Show that the automorphism group of 1_{BG} can be identified with the center $Z(G)$ of G.

Representative 2-commutative diagrams will be

$$\begin{tikzcd} \underline{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \underline{\Lambda} \arrow[d] \\ \underline{\Lambda} \arrow[r] & BG \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} BG \arrow[r, "1_{BG}"] \arrow[dr, "1_{BG}"'] & BG \arrow[d, "1_{BG}"] \arrow[dl, phantom, "\Rightarrow" near start] \\ & BG \end{tikzcd}$$

Diagrams such as the first one will appear in the next section. The second diagram actually links up with a more advanced topic, group actions on a stack. A finite group H can act on BG, with every $h \in H$ acting by the identity map 1_{BG}, so that the "quotient" is the classifying stack of a group which is a nontrivial extension of H by G. (This will be a sort of quotient that generalizes how H can act on a point with stack quotient $[\bullet/H] = BH$.) In fact we get precisely the extensions which classically are classified by group cohomology $H^2(H, Z(G))$. The point is that the usual condition for a group action $x \cdot (hh') = (x \cdot h) \cdot h'$ is replaced by 2-commutative diagrams with a further requirement on the 2-morphisms $\Rightarrow$, and these will amount to the cocycle condition of group cohomology.

5. Fiber products of CFGs

We come to an important construction of CFGs, the fiber product. We will have *2-cartesian diagrams* which, just as for schemes, are diagrams which express the fact that one CFG is isomorphic to the fiber product of a pair of CFGs over a third CFG. Also, as in the usual setting, there will be a universal property characterizing such diagrams. However, this property relies heavily upon the notion of 2-morphism that has just been introduced. So for this reason we prefer to give a direct construction of the fiber product of CFGs, which will satisfy a "strict" universal property, and to define 2-cartesian diagrams as those involving a CFG that is isomorphic to "the" fiber product of the given CFGs. Afterwards, we will give the universal property as an optional remark.

Definition 2.24. Given $\mathfrak{X}$, $\mathfrak{Y}$, and $\mathfrak{Z}$, all CFGs over $\mathcal{S}$, and morphisms $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$, the **fiber product** $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is the category whose objects are triples (x, y, α), where x is an object in $\mathfrak{X}$, y is an object in $\mathfrak{Y}$ (over the same S in $\mathcal{S}$), and α is an isomorphism from $f(x)$ to $g(y)$ in $\mathfrak{Z}$ (over the identity on S). A morphism from (x', y', α') to (x, y, α) is given by morphisms $x' \to x$ in $\mathfrak{X}$ and $y' \to y$ in $\mathfrak{Y}$ (over the same morphism $S' \to S$ in $\mathcal{S}$), such that the diagram

$$\begin{tikzcd} f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x) \arrow[d, "\alpha"] \\ g(y') \arrow[r] & g(y) \end{tikzcd}$$

commutes. Compositions of morphisms are defined in the obvious way, and there is an obvious projection from $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ to $\mathcal{S}$.

We have two canonical projections p and q from $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ to $\mathfrak{X}$ and $\mathfrak{Y}$. There is a 2-commutative diagram

$$\begin{tikzcd} & \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathfrak{X} \arrow[dr, "f"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathfrak{Y} \arrow[dl, "g"] \\ & \mathfrak{Z} & \end{tikzcd}$$

where $\Rightarrow$ indicates a 2-morphism $f \circ p \Rightarrow g \circ q$. This 2-morphism is given by α: for an object $\xi = (x, y, \alpha)$ in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$, we have $f \circ p(\xi) = f(x)$ and $g \circ q(\xi) = g(y)$, so α is an isomorphism of $f \circ p(\xi)$ with $g \circ q(\xi)$.

Example 2.25. Here are some examples of fiber products of CFGs.

(1) If X, Y, and Z are objects and $X \to Z$ and $Y \to Z$ are morphisms in $\mathcal{S}$, then

$$\underline{X} \times_{\underline{Z}} \underline{Y} \cong \underline{X \times_Z Y}.$$

Indeed, the fibers over any object S of both sides are sets, and the bijection between them is a result of the usual universal property of the fiber product.

(2) Recall that $\underline{\Lambda}$ is simply the base category $\mathcal{S} = (\mathrm{Sch}/\Lambda)$. The *product* of $\mathfrak{X}$ and $\mathfrak{Y}$ will be $\mathfrak{X} \times_{\underline{\Lambda}} \mathfrak{Y}$. An object is an object of $\mathfrak{X}$ and an object of $\mathfrak{Y}$ (over the same object S in $\mathcal{S}$). A morphism is a morphisms in $\mathfrak{X}$ and a morphism in $\mathfrak{Y}$ (over the same morphism in $\mathcal{S}$). This product will also be denoted $\mathfrak{X} \times \mathfrak{Y}$. In case $\mathfrak{Y} = \mathfrak{X}$ we have a *diagonal morphism* $\Delta_{\mathfrak{X}} \colon \mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$, sending an object s to (s, s) and a morphism φ to (φ, φ).

(3) Let G be an algebraic group. Consider a morphism $E \colon \underline{S} \to BG$ corresponding (Proposition 2.20) to a G-torsor $E \to S$. There is also the morphism $\mathrm{triv} \colon \underline{\Lambda} \to BG$ which assigns to every scheme T the trivial G-torsor $T \times G \to T$. The fiber product $\underline{S} \times_{BG} \underline{\Lambda}$, we claim, is isomorphic to $\underline{E}$. This fact will be expressed by saying there is a 2-cartesian diagram

$$\begin{tikzcd} \underline{E} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \underline{\Lambda} \arrow[d, "\mathrm{triv}"] \\ \underline{S} \arrow[r, "E"'] & BG \end{tikzcd}$$

Indeed, an object of the fiber product, over a scheme T, is a morphism $T \to S$ together with a G-equivariant isomorphism of E_T (the given pullback of E to T) with $T \times G$. The identity section $T \to T \times G$ corresponds, via this isomorphism, to a section of $E_T \to T$, giving rise to $T \to E_T \to E$, an object of $\underline{E}$. Conversely, given $T \to E$, we obtain a section of $E_T \to T$ from the fact that E_T is a fiber product of T with E over S. Finally, G-equivariance uniquely determines uniquely the isomorphism $E_T \cong T \times G$.

(4) We had noted in Example 2.9(5) that the forgetful morphism from $\mathcal{M}_{g,1}$ to $\mathcal{M}_g$ can be regarded as the universal curve. This example points out why. Let $C \to S$ be a family of curves of genus g. Then we claim $\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \cong \underline{C}$, i.e., there is a 2-cartesian diagram

$$\begin{tikzcd} \underline{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathcal{M}_{g,1} \arrow[d, "\text{forgetful}"] \\ \underline{S} \arrow[r, "C"'] & \mathcal{M}_g \end{tikzcd}$$

An object over T of the fiber product consists of a morphism $T \to S$, a family of curves $C' \to T$ with section σ, and an isomorphism $\vartheta \colon C_T \cong C'$ over T. The section σ, composed with ϑ and projection to C is a morphism $T \to C$, so we have a map

$$\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \to \underline{C}.$$

A map the other way assigns to $f \colon T \to C$ the family of curves $C_T \to T$ with section induced by f. The composition of these maps in one order is equal to $1_{\underline{C}}$, and in the other order is naturally isomorphic, by ϑ, to $1_{\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}}$.

Exercise 2.4. If each of $\mathfrak{X}$, $\mathfrak{Y}$ and $\mathfrak{Z}$ is a CFG/$\mathcal{S}$, then $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is also a CFG/$\mathcal{S}$. [Hint: show that it satisfies the CFG axioms (1) and (2′) (Exercise 2.2).]

Remark 2.26. In any fiber product of the form $\underline{S} \times_{\mathfrak{X}} \underline{T}$, we get a category that has no nontrivial morphisms. This is because $\underline{S}$ and $\underline{T}$ are categories with no nontrivial morphisms. (By a CFG with "no nontrivial morphisms" we mean one whose fibers are sets, i.e., categories with only identity morphisms. Then, between any two objects t and s there will be precisely one morphism over $f\colon T \to S$ when $t = f^*(s)$ and otherwise no morphism over f.) So, in the discussion in (3) there is no mention of morphisms. Whereas, in (4) there are nontrivial morphisms in the fiber product: as a category, $\underline{S} \times_{\mathcal{M}_g} \mathcal{M}_{g,1}$ is equivalent but not isomorphic to the category $\underline{C}$.

In fact, for any CFG $\mathfrak{X}$ the category $\mathrm{HOM}(\mathfrak{X}, \underline{S})$ is a set. Later we will see instances where, to a CFG $\mathfrak{X}$, we can associate a scheme M, with morphism $\mathfrak{X} \to \underline{M}$, inducing set bijections $\mathrm{HOM}(\mathfrak{X}, \underline{X}) = \mathrm{Hom}_{\mathcal{S}}(M, X)$ for all schemes X. For $\mathfrak{X} = \mathcal{M}_g$ we will be able to take $M = M_g$, the classical moduli space of curves of genus g.

The fiber product satisfies the following **strict universal property**: given maps $u\colon \mathfrak{W} \to \mathfrak{X}$, $v\colon \mathfrak{W} \to \mathfrak{Y}$, together with a natural isomorphism $f \circ u \Rightarrow g \circ v$, there is a unique map $(u, v)\colon \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ with $p \circ (u, v) = u$ and $q \circ (u, v) = v$, so that the natural isomorphism from $f \circ u$ to $g \circ v$ is the one determined by $f \circ p \Rightarrow g \circ q$ (by the identities $f \circ u = f \circ p \circ (u, v) \Rightarrow g \circ q \circ (u, v) = g \circ v$).

Notice that strict universal property involves maps to $\mathfrak{X}$ and $\mathfrak{Y}$ and a 2-morphism of the composite maps to $\mathfrak{Z}$. These are precisely the data to determine a 2-commutative diagram

(1)
$$
\begin{tikzcd}
 & \mathfrak{W} \arrow[dl, "u"'] \arrow[dr, "v"] & \\
\mathfrak{X} \arrow[dr, "f"'] \arrow[rr, phantom, "\Rightarrow"] & & \mathfrak{Y} \arrow[dl, "g"] \\
 & \mathfrak{Z} &
\end{tikzcd}
$$

Now we say that the diagram (1) is **2-cartesian** (or is a fiber diagram, or a pullback diagram) if the morphism $(u, v)\colon \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ determined by the strict universal property, is an isomorphism of CFGs. We have met some 2-cartesian diagrams in Example 2.25. Here is one more.

Exercise 2.5. Given a CFG $\mathfrak{X}$ over $\mathcal{S}$, define the *inertia CFG* to be the following category, which will be denoted $I_{\mathfrak{X}}$. An object of $I_{\mathfrak{X}}$ is a pair (s, σ) where s is an object of $\mathfrak{X}$ (over some S in $\mathcal{S}$), and σ is an isomorphism $s \to s$ over 1_S. A morphism $(s', \sigma') \to (s, \sigma)$ is a morphism $s' \to s$ (over some $f\colon S' \to S$) such that $f^*(\sigma) = \sigma'$. There is a functor $i\colon I_{\mathfrak{X}} \to \mathfrak{X}$ which forgets σ.

(i) $I_{\mathfrak{X}}$ is a CFG.

(ii) There is a 2-cartesian diagram

$$\begin{tikzcd} I_{\mathfrak{X}} \arrow[r, "i"] \arrow[d, "i"'] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{X} \arrow[d, "\Delta_{\mathfrak{X}}"] \\ \mathfrak{X} \arrow[r, "\Delta_{\mathfrak{X}}"'] & \mathfrak{X} \times \mathfrak{X} \end{tikzcd}$$

(iii) Let $\Lambda = \mathrm{Spec}(k)$ where k is an algebraically closed field, and let G be a finite group. For $\mathfrak{X} = BG$, we have $I_{\mathfrak{X}} \cong [G/G]$ where G acts on itself by conjugation.

The next two exercises gather some basic facts about fiber products which are familiar from the cas of schemes (or objects of a general category).

Exercise 2.6. Given morphisms $\mathfrak{X} \to \mathfrak{Y}$, $\mathfrak{Y} \to \mathfrak{Z}$, and $\mathfrak{W} \to \mathfrak{Z}$, construct an isomorphism of CFGs (equivalence of categories) between $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})$ and $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}$.

Exercise 2.7. Given $f\colon \mathfrak{X} \to \mathfrak{Z}$ and $g\colon \mathfrak{Y} \to \mathfrak{Z}$, we have morphisms $f \times g\colon \mathfrak{X} \times \mathfrak{Y} \to \mathfrak{Z} \times \mathfrak{Z}$ and a diagonal morphism $\mathfrak{Z} \to \mathfrak{Z} \times \mathfrak{Z}$. Construct an isomorphism of CFGs

$$\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \cong (\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}.$$

The corresponding 2-cartesian diagrams are

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{W} \arrow[d] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Z} \arrow[d] \\ \mathfrak{X} \times \mathfrak{Y} \arrow[r] & \mathfrak{Z} \times \mathfrak{Z} \end{tikzcd}$$

Example 2.27. Suppose we are given morphisms $\mathfrak{X} \to \mathfrak{U} \leftarrow \mathfrak{Y} \to \mathfrak{V} \leftarrow \mathfrak{Z}$ of CFGs over $\mathcal{S}$. Define a category $\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}$ whose objects are (x, y, z, α, β), with x, y, z objects in $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$, respectively, over some S in $\mathcal{S}$; α is a map from the image of x to the image of y in $\mathfrak{U}$, and β is a map from the image of y to the image of z in $\mathfrak{V}$, all over 1_S. Morphisms are defined as in the case of fiber products. Then we have isomorphisms (these are, in fact, isomorphisms of categories)

$$\mathfrak{X} \times_{\mathfrak{U}} (\mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}) \cong \mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \cong (\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y}) \times_{\mathfrak{V}} \mathfrak{Z}.$$

Projections to $\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y}$ and to $\mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z}$ give rise to a diagram

$$\begin{tikzcd}
\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \arrow[r] \arrow[d] & \mathfrak{Y} \times_{\mathfrak{V}} \mathfrak{Z} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Z} \arrow[d] \\
\mathfrak{X} \times_{\mathfrak{U}} \mathfrak{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & \mathfrak{Y} \arrow[r] \arrow[d] & \mathfrak{V} \\
\mathfrak{X} \arrow[r] & \mathfrak{U} &
\end{tikzcd}$$

The upper-left square (which, in fact, strictly commutes) is 2-cartesian; this can be seen by an application of Exercise 2.6.

REMARK 2.28. The strict universal property characterizes $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ up to an isomorphism of categories. This is too strict: the natural notion of isomorphism of CFGs is an equivalence of categories. There is a more natural universal property, which we should emphasize is not required for the material in this book. A diagram (1) is 2-cartesian if and only if the following universality condition is satisfied. Given a CFG $\mathfrak{U}$, define a category

$$\mathrm{HOM}\left(\mathfrak{U}, \begin{array}{ccc} & & \mathfrak{Y} \\ & & \downarrow \\ \mathfrak{X} & \rightarrow & \mathfrak{Z} \end{array}\right)$$

whose objects are triples (m, n, δ) where $m\colon \mathfrak{U} \to \mathfrak{X}$ and $n\colon \mathfrak{U} \to \mathfrak{Y}$ are morphisms, and δ is a 2-morphism from $f \circ m$ to $g \circ n$. A morphism from (m, n, δ) to (m', n', δ') will consist of a pair of 2-morphisms $m \Rightarrow m'$ and $n \Rightarrow n'$ such that the composite 2-morphism $f \circ m \Rightarrow g \circ n \Rightarrow g \circ n'$ is equal to the composite $f \circ m \Rightarrow f \circ m' \Rightarrow g \circ n'$. There is a functor

$$\mathrm{HOM}(\mathfrak{U}, \mathfrak{W}) \longrightarrow \mathrm{HOM}\left(\mathfrak{U}, \begin{array}{ccc} & & \mathfrak{Y} \\ & & \downarrow \\ \mathfrak{X} & \rightarrow & \mathfrak{Z} \end{array}\right)$$

which sends $h\colon \mathfrak{U} \to \mathfrak{W}$ to $(u \circ h, v \circ h, f \circ u \circ h \Rightarrow g \circ v \circ h)$ and $h \Rightarrow h'$ to the pair consisting of $u \circ h \Rightarrow u \circ h'$ and $v \circ h \Rightarrow v \circ h'$. The universality condition is that this functor should be an equivalence of categories for any CFG $\mathfrak{U}$.

Answers to Exercises

2.1. If one constructs a principal GL_n-bundle by means of transition functions, the vector bundle is constructed from the same transition functions. The same idea works in (6). In both cases, note that G is the automorphism group of the fiber.

2.2. To prove (2), chose by (1) some $\gamma_0\colon u \to t$ over g. Using (2′) for h, one obtains $\theta\colon u \to u$ over 1_U with $\eta = \varphi \circ \gamma_0 \circ \theta$. Then $\gamma = \gamma_0 \circ \theta$ is a solution. If γ and γ' were two solutions, applying (2′) to the morphism g, one finds $\tau\colon u \to u$ over 1_U with $\gamma' = \gamma \circ \tau$. Since $\varphi \circ \gamma \circ \tau = \varphi \circ \gamma$, the uniqueness for maps over h implies that $\tau = 1_u$.

2.3. For (i), by Proposition 2.20 this is the automorphism group of the trivial G-torsor G over Λ. This is G. Directly, $g_0 \in G$ corresponds to the automorphisms $(s,g) \mapsto (s, g_0 g)$ of $S \times G$, for arbitrary S. For (ii), consider a 2-morphism $\alpha\colon 1_{BG} \to 1_{BG}$, that is, a specification of automorphisms of G-torsors $E \to S$ compatible with the morphisms in BG. Restricted to trivial G-torsors $S \times G$, these must be of the form $(s,g) \mapsto (s, g_0 g)$ for some $g_0 \in G$. But every G-torsor is locally trivial, so α is completely determined by g_0, and it remains to see that g_0 is constrained to lie in the center $Z(G)$. For the trivial G-torsor G over Λ, the automorphism corresponding to g_0 is $g \mapsto g_0 g$. For any $h \in G$ we have an isomorphism in BG sending G to G by $g \mapsto hg$. Compatibility forces $hg_0 g = g_0 hg$ for any $h \in G$, i.e., $g_0 \in Z(G)$. By descent, any $g_0 \in Z(G)$ determines an automorphism of an arbitrary G-torsor $E \to S$.

2.4. Given $h\colon T \to S$ and an object $s = (x, y, \alpha)$ in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ over S, to find a morphism $t \to s$ over h, choose $x' \to x$ and $y' \to y$ over h, and use Axiom (2) for $\mathfrak{Z}$ to find a morphism $\alpha'\colon f(x') \to g(y')$ over 1_T so that the diagram

$$\begin{tikzcd} f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x) \arrow[d, "\alpha"] \\ g(y') \arrow[r] & g(y) \end{tikzcd}$$

commutes. Then we have $t = (x', y', \alpha') \to s$ over h. To prove Axiom (2′), suppose we have $(x, y, \alpha) \to (x_0, y_0, \alpha_0)$ and $(x', y', \alpha') \to (x_0, y_0, \alpha_0)$ over h. This means we have $x \to x_0$ and $x' \to x_0$ in $\mathfrak{X}$, $y \to y_0$ and $y' \to y_0$ in $\mathfrak{Y}$, all over h, and a commutative diagram

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d, "\alpha_1"'] & f(x_0) \arrow[d, "\alpha"'] & f(x') \arrow[l] \arrow[d, "\alpha_2"'] \\ g(x) \arrow[r] & g(x_0) & g(x') \arrow[l] \end{tikzcd}$$

in $\mathfrak{Z}$, with the horizontal maps over h. From (2′) for $\mathfrak{X}$ and $\mathfrak{Y}$ we get morphisms $x \to x'$ and $y \to y'$. We need to know that the left square in the diagram

$$\begin{tikzcd} f(x) \arrow[r] \arrow[d, "\alpha"'] & f(x') \arrow[r] \arrow[d, "\alpha'"'] & f(x_0) \arrow[d, "\alpha"'] \\ g(x) \arrow[r] & g(x') \arrow[r] & g(x_0) \end{tikzcd}$$

commutes. This follows from the fact that the large rectangle commutes, and the uniqueness in $\mathfrak{Z}$ of maps from $f(x)$ to $f(x')$ over 1_T with given maps to $g(x_0)$ over h.

2.5. For (i), if $\varphi\colon t \to s$ is a morphism in $\mathfrak{X}$ over $f\colon T \to S$ then we have $(t, f^*(\sigma)) \to (s, \sigma)$ in $I_{\mathfrak{X}}$ over f. Let, now, $g\colon U \to T$ be a morphism, $h = f \circ f$, and morphisms $\varphi\colon (t, \tau) \to (s, \sigma)$ and $\eta\colon (u, \upsilon) \to (s, \sigma)$ in $I_{\mathfrak{X}}$ over f and h, respectively. Axiom (2) dictates a unique morphism $\gamma\colon u \to t$ in $\mathfrak{X}$. Since $\gamma^*(\tau) = \gamma^*(\varphi^*(\sigma)) = \eta^*(\sigma) = \upsilon$, we have $\gamma\colon (u, \upsilon) \to (t, \tau)$ in $I_{\mathfrak{X}}$. For (ii), we have a 2-morphism $\Delta_{\mathfrak{X}} \circ i \Rightarrow \Delta_{\mathfrak{X}} \circ i$, $(s, \sigma) \mapsto \sigma \times 1_s\colon (s, s) \to (s, s)$, hence a morphism $I_{\mathfrak{X}} \to \mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}$. A map the other way is given by $(s, s', \sigma \times \sigma') \mapsto (s, \sigma'^{-1} \circ \sigma)$. One composition is the identify $1_{I_{\mathfrak{X}}}$. The other composition is naturally isomorphic to $1_{\mathfrak{X} \times_{\mathfrak{X} \times \mathfrak{X}} \mathfrak{X}}$ by $(s, s', \sigma \times \sigma') \mapsto [1_s \times \sigma' \colon (s, s, \sigma'^{-1} \circ \sigma, 1_s) \to (s, s', \sigma \times \sigma')]$. For (iii), define $I_{BG} \to [G/G]$ by sending $(E \to S, \sigma \colon E \to E)$ to the torsor $E \to S$ together with map $E \to G$ which sends $e \in E$ to the unique $g \in G$ such that $\sigma(e) = e \cdot g$. This is an isomorphism of categories.

2.6. There is a morphism $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}$ sending $(x, (y, z, \beta), \alpha)$ to $(x, z, \beta \circ g(\alpha))$, where g denotes the morphism $\mathfrak{Y} \to \mathfrak{Z}$, and a morphism $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \to \mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})$ sending (x, z, γ) to $(x, (f(x), z, \gamma), 1_x)$, with f the morphism $\mathfrak{X} \to \mathfrak{Y}$. One composition is $1_{\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W}}$, while the other composition is naturally isomorphic to $1_{\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W})}$ by the pair consisting of the identity of the morphism $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X}$ and the natural isomorphism from $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{W} \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}$ to $\mathfrak{X} \times_{\mathfrak{Y}} (\mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}) \to \mathfrak{Y} \times_{\mathfrak{Z}} \mathfrak{W}$ given by $(x, (y, z, \beta), \alpha) \mapsto [(\alpha, 1_z) \colon (f(x), z, \beta \circ g(\alpha)) \cong (y, z, \beta)]$.

2.7. To an object (x, y, α) in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$, assign the object $((x, y), g(y), \alpha \times 1_{g(y)})$ in $(\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}$. To an object $((x, y), z, \alpha \times \beta)$ in $(\mathfrak{X} \times \mathfrak{Y}) \times_{\mathfrak{Z} \times \mathfrak{Z}} \mathfrak{Z}$, assign the object $(x, y, \beta^{-1} \circ \alpha)$ in $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$. As in the previous exercise, the composition of these morphisms in one order is identity, and in the other order is naturally isomorphic to identity.